Simple Heuristic Approach To Introduction Of The Black-Scholes Model

Rossitsa Yalamova, University of Lethbridge, Canada

ABSTRACT

A heuristic approach to explaining of the Black-Scholes option pricing model in undergraduate classes is described. The approach draws upon the method of protocol analysis to encourage students to `think aloud' so that their mental models can be surfaced. It also relies upon extensive visualizations to communicate relationships that are otherwise inaccessible at the average student's level of mathematical sophistication. This paper presents visual illustration of the changes in the probability measures with concrete examples breaking the option premium into four different components. The relationship between changes in variables and those components are graphically and algebraically illustrated.

Keywords: option pricing, components of the premium, visualization

INTRODUCTION

Two fundamental assumptions that underlie formal education systems are that students (1) retain knowledge and skills they acquire in school, and (2) can apply them in situations outside the classroom. If we structure teaching and testing techniques in a way to allow rote learning, we might achieve certain retention of "knowledge" while the skills acquired might not be useful for a career progress. Moreover, memorizing would not help in a problem solving situations and various other applications outside the classroom. Problem solving is fundamental to learning finance and especially crucial in such quantitative areas as derivatives pricing. This note is aimed at the instructional design and educational technology for one of the main topics of derivative valuation that sets the foundations for understanding the basic principles of risk management and hedging. In problem interpretations students are believed to find mapping between the problem and a pre-existing schema. When derivatives are introduced for the first time at undergraduate level they might be assigned to the wrong schema for the lack of relevant existing one. Aspects of the problem that are inconsistent with the assigned schema could be ignored. Moreover, for the development of the instructional design in this quantitative field, I used protocol analysis, a method for qualitative research in educational psychology. I encourage students to think loud while solving a math problem. This indicates which information the subject is attending to, but is explicitly not interpreted as an explanation or justification for behavior. This information may be used in the instructional design at the level of understanding the problem. On the other hand, verbal analysis using learners' explanations to reveal their mental model or misconceptions may give cues for directing them towards devising a plan towards the application, helping them to find the connection between the data and the unknown in order to avoid the choice of wrong schema. As a result we develop instructional specifications to build a schema in the long-term memory that will be supporting the centrality of analogical thinking to problem solving.

PROBLEM-SOLVING APPROACH

The mental processes between stimulus and response determine how people understand, diagnose, and solve problems. Algorithms do not necessarily lead to comprehension but promise a solution; while heuristics are understood but do not always guarantee solutions.

The Black-Scholes model of option pricing requires knowledge of stochastic calculus and solution of partial deferential equations that is beyond the scope of undergraduate management course. Simply substituting the values of the variables in the formula (algorithm) not only defies the purpose of teaching but also inhibits students' analytical thinking and interest in the quantitative risk modeling area. Therefore, introducing the model at undergraduate level creates a challenge for intuitive explanation with emphasis on graphical representation to enhance the learning experience, as George Pólya in "How to Solve It" suggests. Pólya's heuristic approach also proposes to work the problem backwards, assuming that we already have a solution. This method allows working out the solution set of the option payoffs and then shifting to the paradigm of the discount cash flow valuation of financial assets.

VISUALIZATION HEURISTICS

Graphs, drawings and other visualization tools in the instructional materials are part of the development of a delivery system aimed to meet specific learning needs. We start our heuristic approach from the graph of the payoff for a call option at expiration. We spent the first part of the introductory derivatives course teaching option payoffs with given premiums and striking prices. Graphical representation facilitates the analysis and provides intrinsic value solution for a range of stock prices at expiration. Although, the geometric solution seems to be preferred by the majority of the students, I always provide the algebraic and "table" type solution, which is more of a scenario analysis with incremental stock price algebraic solutions. Graphs also provide visual interpretation of the call/put parity condition as well as the risk less hedge that underlines the model. Strong (2004) provides very good examples on that account.

Understanding the payoff of an option before introducing the Black-Scholes pricing model is essential as this payoff is the future cash flow that determines the price of the asset today. Discounted future cash flow valuation is not new for the students and the risk-neutral framework even makes it simpler defining the discount factor equal to the risk free rate. This simplifies the problem to finding the intrinsic value $C = S - K$ of the option at expiration and discounting it to today's value (price) at the risk free rate.

On a graph the intrinsic value of an option is the distance between the stock price S and the striking price K, visualization also reinforces the relationship between premium size of options on the same stock with the same expiration but different striking prices $C_1 > C_2$, while $K_1 < K_2$ (note that time value and non-linearity in risk break this straight relationship).

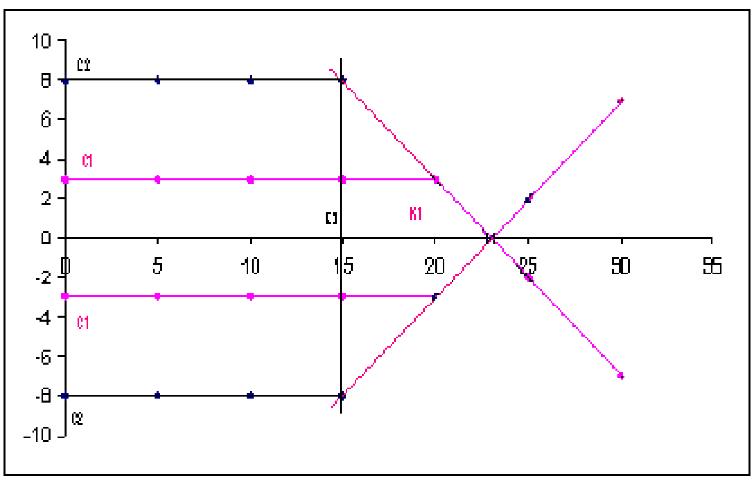

Figure1: Call Option Pay-off – K_1>K_2 and C_1<C_2.

In my instruction model, I attempt to draw on the strength of visualization in connecting the changes of variables with the payoffs for the two counterparties in a call option trade. I also give rise to intuition about the variable changes and their effect on the option value.

Financial asset pricing is based on discounted future cash flows. In this sense the Black-Scholes model is no different. Option is priced in risk free framework based on risk less hedge with a long share position and N short calls. This automatically solves the problem of choosing the appropriate discount factor and as the option is

European type with no dividends we have to worry only about the payoff (or intrinsic value) at expiration $C = \max(S - K; 0)$ then discount that cash flow at the risk free rate. The focal point of the method is determining the stock price on that particular day when the payoff is received. Therefore, we need the best estimate for the stock price at expiration to calculate the value for the option today. Knowing the stock price today we have no way to predict the stock price in the future. Based on Efficient market and Brownian motion modeling of stock prices, the best estimate of tomorrow's price is today's price. The BSOPM uses probabilistic approach that takes into account variability of the stock price and change of this variability with time.

REVIEW OF PROBABILITY AND CONTINUOUS COMPOUNDING/DISCOUNTING CALCULATIONS

According to the model, stock prices follow Brownian motion. The geometric Brownian motion assumes that returns on a stock are normally distributed and the standard deviation of this distribution can be estimated from historical data. This is a key ingredient in the Black-Scholes formula. The standard deviation is also called volatility. The more volatile a stock's price, the less precise the information about its future price and the less our ability to predict or pinpoint where it will be on a future date. We'll be calculating probability of returns in the model, therefore it is important to review the standard normal distribution and find the probability of being smaller or larger than an observation drawn from $N(\mu, \sigma)$.

From statistics class, students are familiar with a normal distribution and can relate the return volatility to the standard deviation. Normal distributions with different means μ and standard deviations σ are normalized:

$$Z = \frac{(x - \mu)}{\sigma}$$

The Z-score defines the probabilities associated with certain observation. For call pricing we always look for the probabilities of prices smaller than the calculated Z-score for the observed value x.

The model eliminates discreteness in price movements and trading time. In the introductory/review part of the class we also explain and assign homework to calculate discrete $r_1 = \frac{P_2 - P_1}{P_1}$ and continuously compounded $r_1 = \ln\left(\frac{P_2}{P_1}\right)$ returns on a daily closing stock price series as part of an Excel exercise. The review of continuous compounding e^{rT} and discounting e^{-rT} also provided the students with the basic calculation skills, removing confusion on the exponential (continuous discounting of the striking price at the risk free rate) and the logarithmic (intrinsic value as instantaneous return) functions in the Black-Scholes pricing formula.

INTRODUCING OPTION PRICING

The Black Scholes Option pricing model is a continuous time model that is introduced as a natural extension of the discrete time binomial pricing. The intuition behind this transition can be presented in the following example: If we divide the periods in the binomial model into smaller parts we are going to get an increasingly larger number of possible end period stock prices and the probability space will be divided into smaller parts. If we do this repetitively we can go to the limit of 0 length periods or continuous prices with a very fine division of the probability space. I then note to students that there are a greater number of paths leading to the middle range prices and only single paths leading to each of the two extreme prices. Intuitively if the up and down branch are symmetric, we will get a normal distribution of the prices at the end period.

Actually, prices do not move symmetrically up and down and do not assume negative value. Therefore, prices are modeled by lognormal distribution, while returns follow normal distribution. Lognormal distribution might be still unfamiliar for undergraduate students, but they have to learn that the lognormal plays an important role in finance. This model, of log returns being normal and corresponding prices being lognormal, is one of the most ubiquitous models in finance. In the Black Scholes formula we apply this property of normal distribution of the log returns.

We can use this approach to visualize the stock price generating process and connect to striking price, volatility, time to expiration and risk free rate. The binomial pricing model and the transition to continuous time Black-Scholes option pricing model will be examined in a separate note.

OPTION PAY-OFF AND STRIKE PRICE

If there are mispriced securities on the market, there are arbitrage opportunities, i.e. making money without taking risk. Looking at the payoff graph (Figure1) and assessing the option premium in terms of arbitrage existence, we may want to break it into two parts for simplification: (1) "profit for the buyer" if the stock price is higher than $K + C$ and (2) "profit for the seller" if the stock price is lower than $K + C$. In other words the buyer would like the option to expire in the money, i.e. above the breakeven ($K + C$) stock price. A buyer incurs limited losses when a stock price is lower than $K + C$ at expiration and to reduce the set of outcome with losses, he tries to reduce the premium C.

Thus we can construe from the graph that for the seller at stock prices larger than the striking price, profits decline, while at stock prices above $K + C$, losses increase. In order to reduce the probability of losses, the seller may try to augment the distance between 0 and $K + C$. Since K is already fixed, the only way to decrease the probability of losses is to increase C, i.e. by charging a higher premium. A seller thus avoids expiration at a stock price higher than $K + C$ by increasing C.

This "optimization" strategy also does not contradict the intuitive logic for a buyer willing to pay less and a seller trying to charge a higher price. It is expected that the market should clear at the "breakeven" point $K + C$. Looking at this differently, as nobody makes "easy" money, (i.e. no arbitrage opportunity), there will be no increase in demand by either position that would lead to disequilibrium in the market and a change in the price.

From the payoff graph (Fig.1) we can infer that the most probable stock price at expiration should coincide with the break even point of "no profit" which should be the case in a risk less position. Roughly, if we are able to pinpoint the stock price at expiration we should be charging a premium equal to the distance to the striking price discounted to today's value.

Summing up, the two counter parties face the same payoff but with opposite signs. It is a zero sum game and each party is trying to find "mispriced" assets to yield a profit. Any mispricing will tend to disappear fast leading to this breakeven point.

This again leads to the realization that if we can give our best estimate on the stock price at expiration, we can work the option premium backwards. Here I point out that the distance between the strike price and the breakeven is equal to the premium on the payoff graph, forming an equilateral triangle (see Fig.2). Once we determine this distance (within certain probability), we have to discount this payoff at the risk free rate for the time between today and expiration (the time when payoff takes place). I then reiterate that if we pay a premium today to secure a risk less position till expiration at time T we are entitled to a risk free return. Conversely if we have calculated the payoff at time T we have to discount it to today to determine the premium we are willing to pay today. This conclusion comes as no surprise given the definition and payoff of European call option on a stock that pays no dividends.

THE BLACK SCHOLES FORMULA

In his 1957 book, George Pólya proposes a heuristic approach to learning that adopts the solution and then works the problem backwards to build a schema in the long memory. This approach is appropriate in the case where most researchers understand how to set the risk-neutral hedge and derive the PDE, but rarely anybody in finance wants to deal with its solution. Even Black and Scholes (1972) skip this intermediate step and propose the solution as a two part formula that calculates the intrinsic and the time value with a probabilistic approach. What I suggest is to break into parts the premium and demonstrate how changes in variables affect the result.

Let us introduce the equation for the call pricing and discuss in more detail the calculation of the Z-scores (d_1 and d_2) and the information about the value of the option the associated probabilities provide. As students are not normally equipped with the knowledge to derive the partial differential equation and are even less likely to be capable of solving it, they should be offered the algorithm with as much heuristic insight as possible:

$$C(S,T) = SN(d_1) - Ke^{-rT}N(d_2)$$

The intrinsic value equation of the call premium is the difference between the stock and the strike price. The pricing model uses today's stock price and, as the strike price will be paid T time later, we discount the striking price at the risk free interest rate - Ke^{-rT}. In addition, there are probability measures in the equations associated with both stock and striking price that actually incorporate measures of the stock price volatility component of the premium. We can also introduce the intuition of the martingale property as the best estimate for the stock price 'tomorrow' is today's price. As there is certain probability distribution associated with this estimate we visualize it on Figure 2.

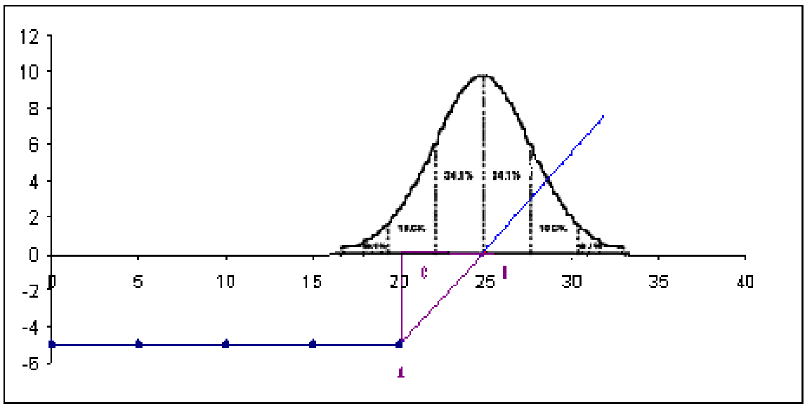

Figure 2: Call Pay-off - The probabilities associated with the stock price at expiration follow normal distribution.

NORMAL DISTRIBUTION AND PROBABILITIES

Our usual assumption is that stock returns are normally distributed with a mean μ and standard deviation σ and we calculate risk (standard deviation) from historical data on stock prices. But as we are basing our model on a risk free hedge, we cannot expect a return on our option equal to the stock return, but only to the risk free rate.

We also assume that the stock price follows a martingale process, i.e., the conditional expected value of the next observation, given all of the past observations, is equal to the last observation. Therefore, we calculate a Z -score (d_1) of the instantaneous option log return $ln\frac{S}{K}$ using today's stock price. Risk measured by the standard deviation σ, changes with time and should be multiplied by $\sqrt{\mathrm{T}}$ (a property of the assumed Brownian motion model of stock prices). Here it is helpful to recall that continuously compounded returns are calculated $r = ln\frac{S}{K}$, while the same return in discrete terms is $r = \frac{S-K}{K}$. The return on the option should have a risk premium with mean zero (risk less hedge) and the Z score computation following the standardization formula should obtain:

$$d_1 = \frac{ln\frac{S}{K} - 0}{\sigma\sqrt{\mathrm{T}}}$$

By adding the risk free rate, we assure that the option return includes the risk free return and the option premium is a hedge against the risky part of the return. The explanation of the last term in the numerator is related to the jagged path of the stock price function. As the stock price is not a smooth function of time, we cannot ignore the second order term expansion of the function, i.e. it is not negligible. (The smoother a function is, the better its change is approximated only with its first derivative). Therefore in other to capture the change of the stock price function (which is the return) we need not to ignore $\frac{\sigma^2}{2}$ term:

$$d_1 = \frac{ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

The probability measure that multiplies the striking price is removed one standard deviation (adjusted for time) to the left and the equation is:

$$d_2 = d_1 - \sigma\sqrt{T}$$

I bring more detail and intuition on these Z-scores (d_1 and d_2) below.

CONCRETE EXAMPLES

Clearly, it is also good advice when developing problem solving skills that if the problem is abstract, then one should periodically introduce concrete examples. I augment my teaching at this point with excel spreadsheet solutions for the Black-Scholes option pricing model that help students to follow the preceding discussion with specific examples.

Call Option at the Money

The first part of the equation involves the stock price S multiplied by a probability measure $N(d_1)$. For an option at the money ($S = K$) the instantaneous return is 0 and if we set the risk free rate at 0, we can calculate the d_1 removed one half standard deviation to the right from zero:

$$d_1 = \frac{ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = \frac{0 + \left(0 + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = \frac{\sigma}{2}\sqrt{T}$$

We may view this as an allowance for the stock price to go up half a standard deviation corrected for time (and still be at the money) until expiration. If T=1, then $\sqrt{T}$ is also one and the d_1-score is simply a half standard deviation away from zero. The higher the standard deviation the higher the probability measure multiplying the stock price.

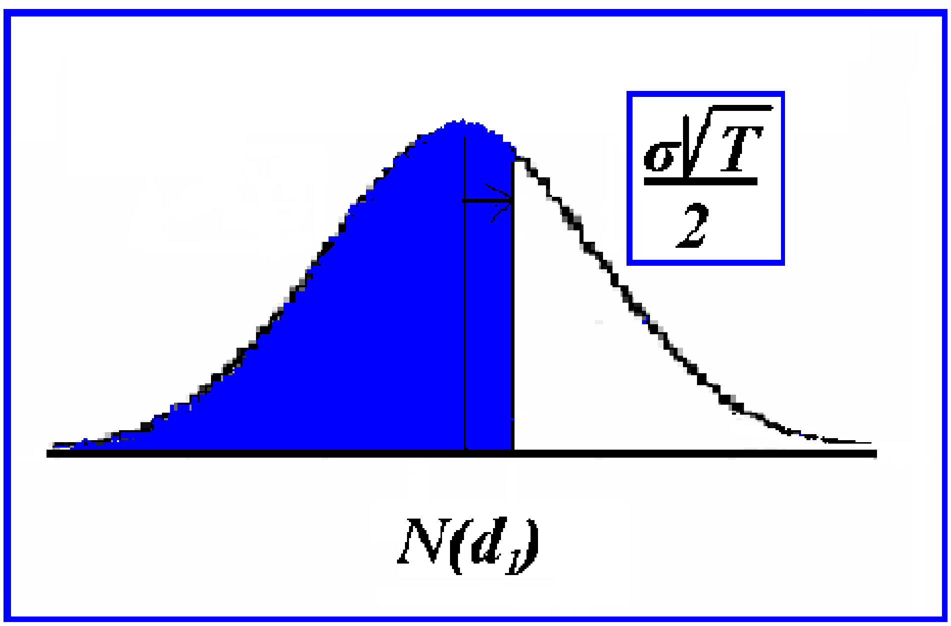

Figure 3: Probability Measure Associated with the Stock Price for an Option at the Money and Risk-Free Rate 0. The Z-score (d_1) is moved a half standard deviation (adjusted for time) to the right.

A stock with a very high risk (volatility) will be multiplied by a higher probability measure. In extreme case the probability measure approaches 1 and the first part of the equation approaches the stock price. Please, note that increase in the probability measure $\mathrm{N}(d_1)$, (e.g. moving d_1 to the right on the real line), will increase the value of the call.

The second part of the equation involves the striking price K discounted to today's value at the risk free rate and another probability measure $\mathrm{N}(d_2)$:

The d_2-score is a standard deviation (adjusted for time) to the left from the d_1-score. If this standard deviation is large (high risk), the probability multiplier for K becomes smaller which makes the call price larger, ceteris paribus. This may explain why extremely volatile stock might be worth buying outright. It is also useful to illustrate the value of the option with the probability space between d_1 and d_2 such that the larger the area, the larger the value of the option.

If we apply the simplified scenario of $\mathrm{S} = \mathrm{K}$ and $\mathrm{r} = 0$, we get:

$$d_2 = d_1 - \sigma\sqrt{\mathrm{T}} = \frac{\sigma}{2}\sqrt{\mathrm{T}} - \sigma\sqrt{\mathrm{T}} = -\frac{\sigma}{2}\sqrt{\mathrm{T}}$$

The option price is a function of the intrinsic value and the probability density between d_1 and d_2, and as the distance between those two scores is related to the standard deviation, we can attest again that the size of the premium is linked to the magnitude of the volatility measure. Simplified case of an option at the money $K = S$ and zero risk free rate places d_1 half standard deviation to the right of the center of the distribution and d_2 half standard deviation to the left of the center.

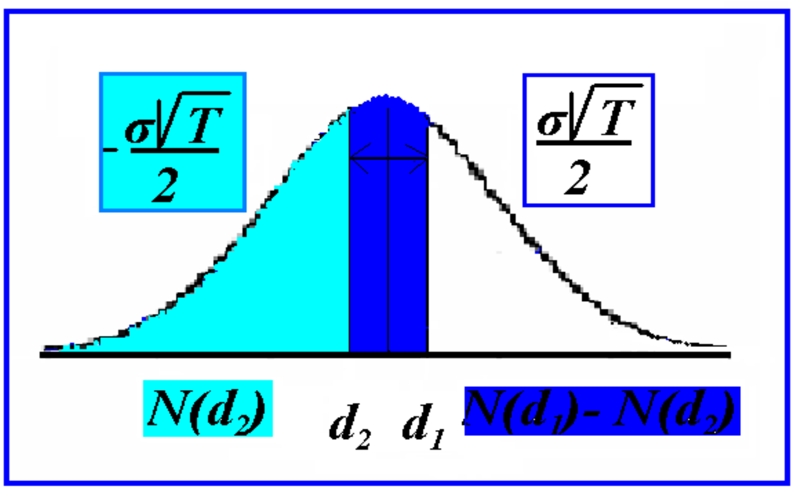

Figure 4: Shaded area between the two Z-scores is associated with the volatility component of the option price.

This also illustrates the impact of the standard deviation on the value of the option and as σ increases $\mathrm{N}(d_1)$ approaches 1, while $\mathrm{N}(d_2)$ approaches 0 and the option price approaches the stock price. On the other extreme an asset with a fixed price has no volatility and option at the money is worth 0.

Call Option at the Money and Positive Risk-Free Rate

Adding a positive term for the risk free rate will also move the d_1-score to the right and increase the probability measure associated with the stock price by increasing the Z-score by $\frac{r}{\sigma\sqrt{\mathrm{T}}}$. Now the probability density determining the time value of the option is enclosed between plus/minus a half standard deviation adjusted for time around $Z = \frac{r}{\sigma\sqrt{\mathrm{T}}}$. Please note the interest rate is normalized.

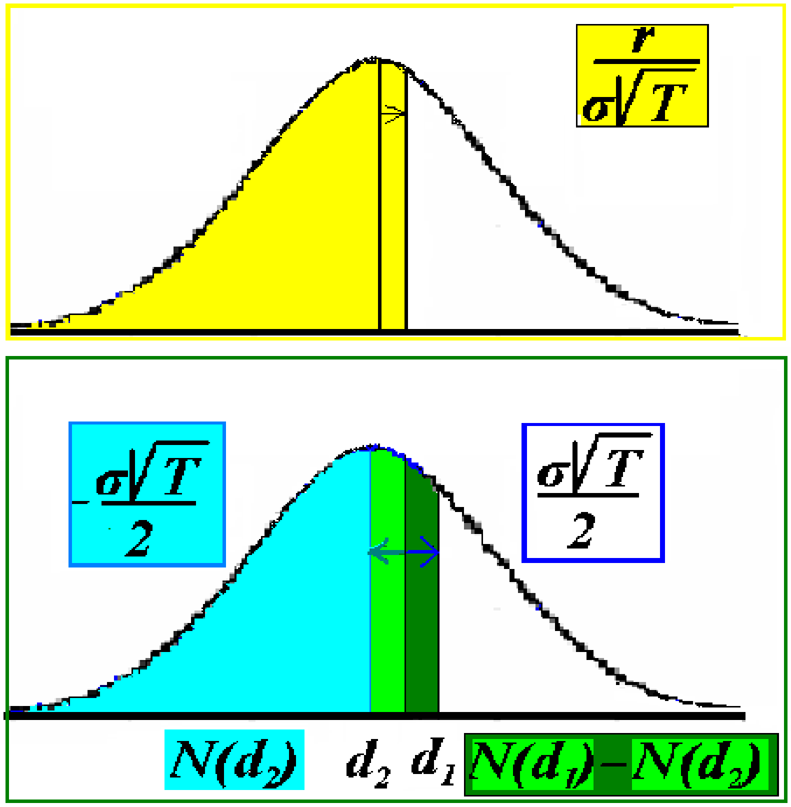

Figure 5: Positive risk free rate moves the shaded area to the right, slightly decreasing its value. Time value of the strike price is the second component of the premium.

We want to discuss the changes in the premium and their origin after we add risk free rate to the option at the money. Note that the distance between the Z-scores remains the same but the area slightly decreases. The change in the premium also contains the time value of the striking price component.

Call Option in the Money and Positive Risk-Free Rate

Gradually adding back the terms in the formula, I then discuss an option with intrinsic value, i.e. one that adds a positive term to the numerator and increases the probability measure. At the maximum for an option with high intrinsic value in a high risk free rate environment and high volatility of underlying stock, the probability measure approaches 1 and the first term approaches the stock price.

We note that an increase in the probability measure $\mathrm{N}(d_1)$, (i.e. moving d_1 to the right on the real line) will increase the probability associated with the stock price, we remind students that this is a measure of the option being in the money.

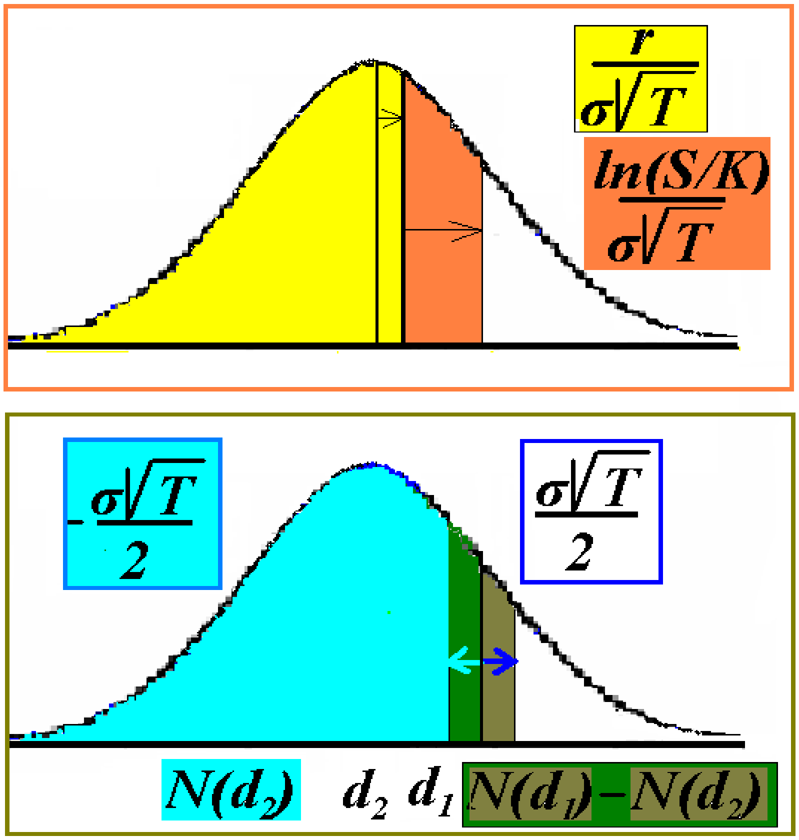

Figure 6: Intrinsic value also moves the shaded area to the right. Although the shaded area decreases futher, another component is added to the premium – intrinsic value.

When we add back intrinsic value and risk free rate in the example, we move the "center point of the shaded area" (which was 0 in the first example) to the right by the amount $\frac{ln\frac{S}{K}}{\sigma\sqrt{T}}$ and $\frac{r}{\sigma\sqrt{T}}$. Then from this new "center" point, we find d_1 a half standard deviation to the right and d_2 a half standard deviation to the left, the distance between them remains constant, but the probability density area is smaller. We added yet another component to the call premium that comprises the intrinsic value multiplied by the probability of being in the money or $N(d_1)$.

The above defined "center point" might go to the left towards the negative Z-scores if the option is out of the money and it is important to consider the out of the money case separately since there are two different effects. The first effect in line with the previous examples is on the probability measures and the second effect will be the impact on the option value in the main equation of the Black Scholes model. Intuitively, we should expect a positive value of the call option, when the volatility component (probability density between d_1 and d_2) cancels out the "negativeness" of the intrinsic value $(S-K)$ and the resulting time value of K when the risk free rate is positive $(S-Ke^{-rT})$.

Call Option out of the Money and 0 Risk-Free Rate

Although, out of the money option has no intrinsic value, it can be priced as the stock price volatility component cancels the "the negative distance between S and K. If the volatility of the stock price is high, the distance between $d_1$and d_2 is large enough and the two multipliers $N(d_1) > N(d_2)$ change the sign of the expression $(S-K)$ from negative to positive. The further out of the money the option is, the smaller the probability density area between d_1 and d_2.

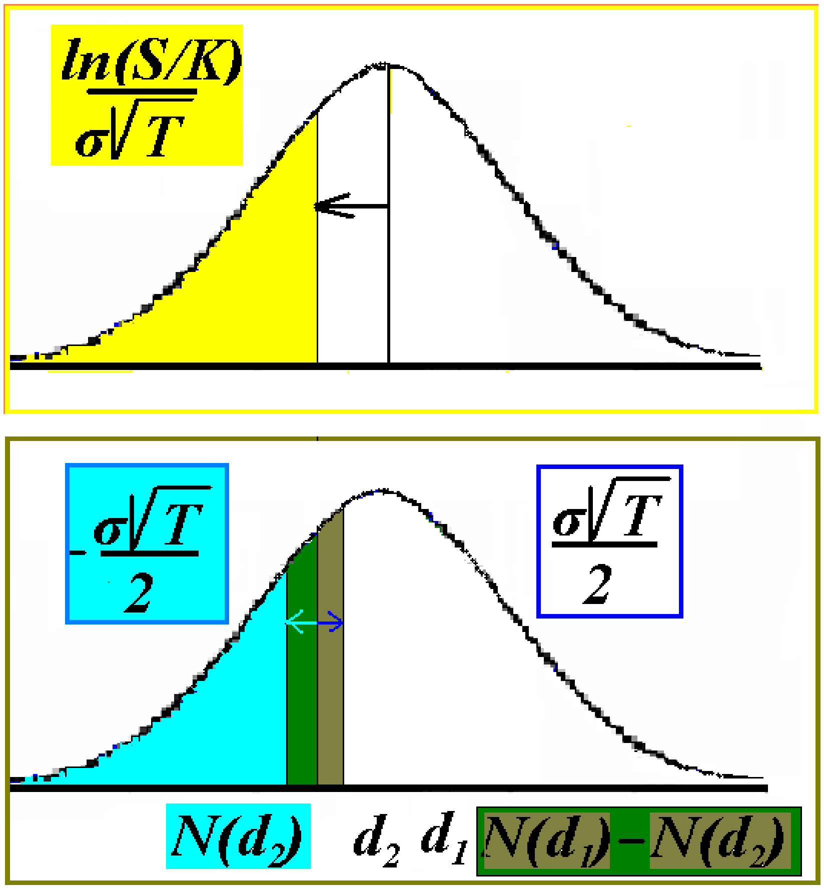

Figure 7: For an option out of the money, the shaded area defining the volatility component of the premium is moved to the left from the center, slightly decreasing its value. On the other side, the negative intrinsic value component decreases the option premium further.

Call option out of the money and positive risk free rate

In this example, the risk free rate increases the probability density area for the volatility component by moving d_1 and d_2 to the right. On the other side, the striking price in the first equation is discounted and that also increases the value of the premium.

It is helpful here to draw attention to the non-linear change in the volatility component of the call premium by pointing at the bell-shape of the standard normal probability density graph. Non-linear changes in the value of the option premium with the changes of the variables are also acknowledged in the non-linear equations for the calculation of the Z-scores, as well as the continuous discounting of the striking price.

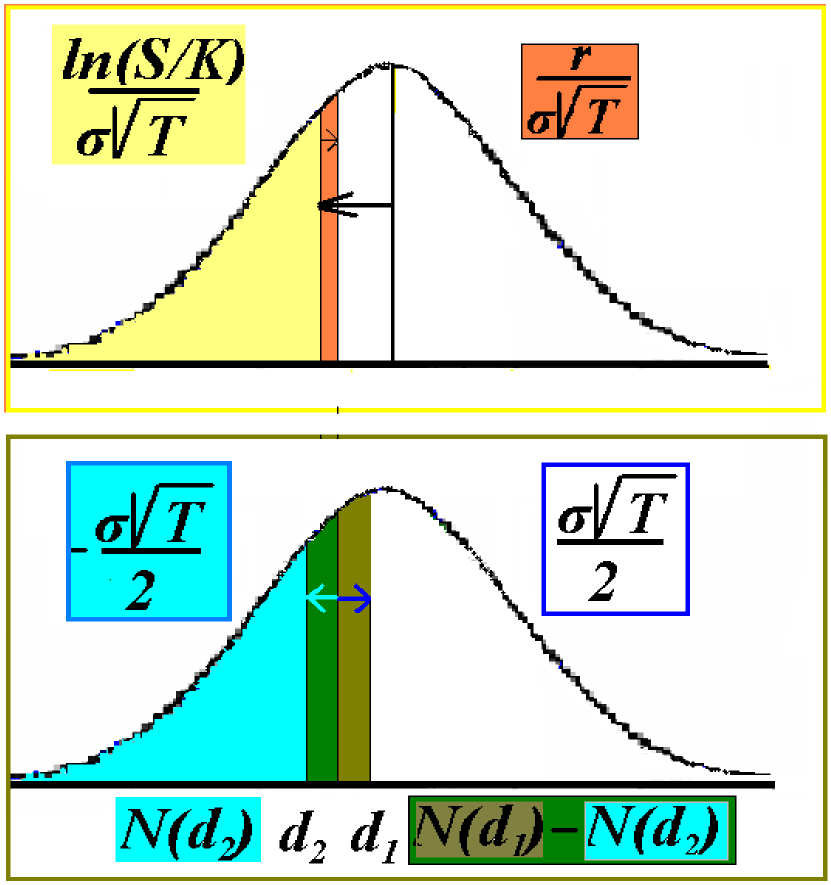

Figure 8: The shaded area of the volatility component for an option out of the money with positive risk free rate moves to the left relative to the previous example. This slightly increases the volatility component, but the option premium is increased also by the time value of the strike price.

SUMMARY

In an undergraduate derivatives course students are usually presented with the solution of the partial differential equation derived by Black and Scholes in their seminal work. In an attempt to avoid rote learning that does not yield improved skill benefits, I have developed a way to present an intuitive explanation of the terms in the equation. I believe that this approach unleashes creative thinking and that the visualizations better illustrate the connections thus allowing students to better grasp the intuitive sense of the model. "What if" scenarios are then explored to test the relationships among variables within the model framework on an auxiliary spreadsheet.

AUTHOR INFORMATION

Rossitsa M. Yalamova is an assistant professor of Finance at the University of Lethbridge in Alberta, Canada and currently a visiting professor in the Facoltà di Scienze Economiche, Università della Svizzera Italiana, Lugano, Switzerland. She holds a Ph.D. in finance from Kent State University. Her research has been published in Fractals, Investment Management and Financial Innovations, International Research Journal of Finance and Economics, and the Asian Academy of Management Journal of Accounting & Finance. She is interested in risk measurement, multifractal models, wavelet analysis of market crashes, phase transition and chaos. She was a participant in the 2007 Santa Fe Institute Complex Systems Summer School held at the Beijing Institute of Theoretical Physics.

REFERENCES

1. Black, F and M. Scholes, The Pricing of Options and Corporate Liabilities, *The Journal of Political Economy*, Vol. 81, No. 3. (May - Jun., 1973), pp. 637-654.
2. G. Polya, *How to Solve It*, 2nd ed., Princeton University Press, 1957
3. Strong, R. (2005), *Derivatives: An Introduction*, 2nd Edition, Thomson South-Western.

NOTES