Min And Max Extreme Interval Values

Marsha L. Jance, Ph.D., Indiana University East School of Business and Economics, USA
Nick T. Thomopoulos, Ph.D., Illinois Institute of Technology Stuart School of Business, USA

ABSTRACT

The paper shows how to find the min and max extreme interval values for the exponential and triangular distributions from the min and max uniform extreme interval values. Tables are provided to show the min and max extreme interval values for the uniform, exponential, and triangular distributions for different probabilities and observation sizes.

Keywords: Min and Max Extreme Interval Values, Min and Max Values

INTRODUCTION

An extreme interval value g_α is an upper bound for a set of data for a specific probability α. For example, given the probability $P(g < g_{0.05}) = 0.05$, $g_{0.05}$ is the extreme interval value and 5% of the data is below $g_{0.05}$.

In the doctoral thesis, *Calculating Min and Max Extreme Interval Values for Various Distributions*, Jance found the min and max extreme interval values for the standard normal, uniform, and exponential distributions for different probabilities α and observation sizes n. In addition, the min and max triangular extreme interval values for different probabilities α and observation sizes n were found by Jance and Thomopoulos. Excel VBA programs were developed to find the min and max extreme interval values and statistics (expected value, standard deviation, median, and mode) for these distributions.

In this paper, it is shown how the min and max extreme interval values for the exponential and triangular distributions are found from the min and max uniform extreme interval values. First of all, some of the min and max uniform extreme interval values are presented. Then it is shown how to find the min and max exponential and triangular extreme interval values from the min and max uniform extreme interval values. Finally, several tables showing the min and max uniform, exponential, and triangular extreme interval values for different observation sizes n and probabilities α are presented.

MIN AND MAX VALUES

Suppose several samples of n observations are taken from a continuous probability distribution (e.g. exponential) with a density function of f(x) and a cumulative distribution function of F(x) and the smallest and largest values for each sample are identified. The smallest observation value (min) and largest observation value (max) will most likely vary from sample to sample of size n. Thus, the min and max values will have a density function and cumulative distribution function.

The min probability density function is $h(g) = nf(g)\left(1 - F(g)\right)^{(n-1)}$ and the max probability density function is $h(g) = nf(g)F(g)^{(n-1)}$ (Hines, Montgomery, Goldsman and Borror 215). For example, given that the exponential density function is $f(x) = \lambda e^{-\lambda x}$ and the cumulative distribution function is $F(x) = 1 - e^{-\lambda x}$, the min exponential probability density function is $h(g) = n(\lambda e^{-\lambda g})\left(1 - \left(1 - e^{-\lambda g}\right)\right)^{(n-1)}$ and the max exponential probability density function is $h(g) = n(\lambda e^{-\lambda g})\left(1 - e^{-\lambda g}\right)^{(n-1)}$ for a given observation size n.

The Excel VBA programs developed by Jance found the min and max extreme interval values by first determining the min and max cumulative distribution function values for the appropriate range and then used interpolation. In addition, the min and max statistics: expected value, standard deviation, median, and mode were found for different observation sizes n.

UNIFORM

Tables 1 and 2 contain some of the min and max uniform extreme interval values found by Jance when the parameters are a = 0 and b = 1. For example when n = 50 and $\alpha = 0.99$, the min uniform extreme interval value is $g_{0.99} = 0.08799$ and the max uniform extreme interval value is $g_{0.99} = 0.99980$. Thus for a sample size of n = 50, 99% of the min values are below 0.08799 and 99% of the max values are below 0.99980.

Table 1: Min Uniform Extreme Interval Values when a = 0 and b = 1 (Jance)

n	α = 0.01	α = 0.10	α = 0.50	α = 0.90	α = 0.99
1	0.01000	0.10000	0.50000	0.90000	0.99000
5	0.00201	0.02085	0.12945	0.36904	0.60189
10	0.00100	0.01048	0.06697	0.20567	0.36904
25	0.00040	0.00421	0.02735	0.08799	0.16824
50	0.00020	0.00211	0.01377	0.04501	0.08799
75	0.00013	0.00140	0.00920	0.03023	0.05956
100	0.00010	0.00105	0.00691	0.02276	0.04501

Table 2: Max Uniform Extreme Interval Values when a = 0 and b = 1 (Jance)

n	α = 0.01	α = 0.10	α = 0.50	α = 0.90	α = 0.99
1	0.01000	0.10000	0.50000	0.90000	0.99000
5	0.39811	0.63096	0.87055	0.97915	0.99799
10	0.63096	0.79433	0.93303	0.98952	0.99900
25	0.83176	0.91201	0.97265	0.99579	0.99960
50	0.91201	0.95499	0.98623	0.99789	0.99980
75	0.94044	0.96977	0.99080	0.99860	0.99987
100	0.95499	0.97724	0.99309	0.99895	0.99990

EXPONENTIAL

An exponential value can be found by using the inverse transform method $x = \frac{-\ln(1-u)}{\lambda}$ where $\lambda > 0$ and u is a uniformly distributed variable with parameters a = 0 and b = 1 (Law and Kelton 460). Thus, the min and max exponential extreme interval values can be found by using the following:

$$g_\alpha = \frac{-\ln(1 - g_{u,\alpha})}{\lambda}$$

where $g_{u,\alpha}$ is the min or max uniform extreme interval value when a = 0 and b = 1 for a particular n and α.

Suppose n = 50, $\alpha = 0.01$, and $\lambda = 1$, then the min exponential extreme interval value is

$$g_\alpha = \frac{-\ln(1 - g_{u,\alpha})}{\lambda} = -\ln(1 - 0.00020) = 0.000200$$

and the max exponential extreme interval value is

$$g_\alpha = \frac{-\ln(1 - g_{u,\alpha})}{\lambda} = -\ln(1 - 0.91201) = 2.430532$$

where $g_{u,\alpha} = 0.00020$ is the min uniform extreme interval value and $g_{u,\alpha} = 0.91201$ is the max uniform extreme interval value for n = 50, $\alpha = 0.01$, a = 0, and b = 1. These values can be found in Tables 1 and 2 respectively.

Tables 3 and 4 show the min and max exponential extreme interval values for different n and α values found using the method: $g_\alpha = \frac{-\ln(1-g_{u,\alpha})}{\lambda}$ where $\lambda = 1$. Some of Jance's original min and max exponential extreme interval values can be found in Tables 5 and 6 respectively for the case $\lambda = 1$. For example when n = 50, $\alpha = 0.01$, and $\lambda = 1$, Table 5 shows that the min exponential extreme interval value is $g_{0.01} = 0.000201$ and Table 6 shows that the max exponential extreme interval value is $g_{0.01} = 2.430532$. One will notice that the min values in Tables 3 and 5 are very close and that the max values in Tables 4 and 6 are very close. The slight differences are due to rounding.

Table 3: Min Exponential Extreme Interval Values when $\lambda = 1$

n	α = 0.01	α = 0.10	α = 0.50	α = 0.90	α = 0.99
1	0.010050	0.105361	0.693147	2.302585	4.605170
5	0.002012	0.021070	0.138630	0.460513	0.921027
10	0.001001	0.010535	0.069318	0.230256	0.460513
25	0.000400	0.004219	0.027731	0.092104	0.184211
50	0.000200	0.002112	0.013866	0.046054	0.092104
75	0.000130	0.001401	0.009243	0.030696	0.061407
100	0.000100	0.001051	0.006934	0.023023	0.046054

Table 4: Max Exponential Extreme Interval Values when $\lambda = 1$

n	α = 0.01	α = 0.10	α = 0.50	α = 0.90	α = 0.99
1	0.010050	0.105361	0.693147	2.302585	4.605170
5	0.507681	0.996850	2.044461	3.870401	6.209621
10	0.996850	1.581482	2.703511	4.558287	6.907755
25	1.782364	2.430532	3.599039	5.470293	7.824046
50	2.430532	3.100871	4.285263	6.161067	8.517193
75	2.820771	3.498920	4.688552	6.571283	8.947976
100	3.100871	3.782751	4.974786	6.858965	9.210340

Table 5: Min Exponential Extreme Interval Values when $\lambda = 1$ (Jance)

n	α = 0.01	α = 0.10	α = 0.50	α = 0.90	α = 0.99
1	0.010050	0.105361	0.693147	2.302585	4.605170
5	0.002010	0.021072	0.138629	0.460517	0.921034
10	0.001005	0.010536	0.069315	0.230259	0.460517
25	0.000402	0.004214	0.027726	0.092103	0.184207
50	0.000201	0.002107	0.013863	0.046052	0.092103
75	0.000134	0.001405	0.009242	0.030701	0.061402
100	0.000101	0.001054	0.006931	0.023026	0.046052

Table 6: Max Exponential Extreme Interval Values when $\lambda = 1$ (Jance)

n	α = 0.01	α = 0.10	α = 0.50	α = 0.90	α = 0.99
1	0.010050	0.105361	0.693147	2.302585	4.605170
5	0.507630	0.996829	2.044470	3.870324	6.210595
10	0.996797	1.581465	2.703559	4.558222	6.903248
25	1.782351	2.430538	3.599221	5.471355	7.819230
50	2.430532	3.100923	4.285463	6.163454	8.512281
75	2.820838	3.498760	4.688620	6.568563	8.917713
100	3.100913	3.782616	4.975151	6.856075	9.205382

TRIANGULAR

A triangular value can be found by using the inverse transform method: $x = \sqrt{cu}$ when $0 \le u \le c$ and $x = 1 - \sqrt{(1-c)(1-u)}$ when $c < u \le 1$ where c is the mode (0 < c < 1) and u is a uniform variable with parameters a = 0 and b = 1 (Law and Kelton 469). The min and max triangular extreme interval values can be found by using the following:

$$g_\alpha = \sqrt{cg_{u,\alpha}} \text{ when } 0 \le g_{u,\alpha} \le c$$

$$g_\alpha = 1 - \sqrt{(1-c)(1-g_{u,\alpha})} \text{ when } c < g_{u,\alpha} \le 1$$

where $g_{u,\alpha}$ is the min or max uniform extreme interval value when a = 0 and b = 1 for a particular n and α.

Suppose n = 50, $\alpha = 0.01$, a = 0, b = 1, and c = 0.50, then the min triangular extreme interval value is

$$g_\alpha = \sqrt{cg_{u,\alpha}} = \sqrt{(0.50)(0.00020)} = 0.01000$$

and the max triangular extreme interval value is

$$g_\alpha = 1 - \sqrt{(1-c)(1-g_{u,\alpha})} = 1 - \sqrt{(1-0.50)(1-0.91201)} = 0.79025$$

where $g_{u,\alpha} = 0.00020$ is the min uniform extreme interval value and $g_{u,\alpha} = 0.91201$ is the max uniform extreme interval value for n = 50, $\alpha = 0.01$, a = 0, and b = 1. These values can be found in Tables 1 and 2 respectively.

Tables 7 and 8 show the min and max triangular extreme interval values for different n and α values found using the above method when the triangular parameters are a = 0, b = 1, and c = 0.50. Tables 9 and 10 contain some of the original min and max triangular extreme interval values found by Jance and Thomopoulos when the triangular parameters are a = 0, b = 1, and c = 0.50. For example when n = 50, $\alpha = 0.01$, a = 0, b = 1, and c = 0.50, Table 9 shows that the min triangular extreme interval value is $g_{0.01} = 0.01002$ and Table 10 shows that the max triangular extreme interval value is $g_{0.01} = 0.79025$. One will notice that the min values in Tables 7 and 9 are very close and that the max values in Tables 8 and 10 are very close. The slight differences are due to rounding.

Table 7: Min Triangular Extreme Interval Values when a = 0, b = 1, and c = 0.50

n	**α = 0.01**	**α = 0.10**	**α = 0.50**	**α = 0.90**	**α = 0.99**
1	0.07071	0.22361	0.50000	0.77639	0.92929
5	0.03170	0.10210	0.25441	0.42956	0.55384
10	0.02236	0.07239	0.18299	0.32068	0.42956
25	0.01414	0.04588	0.11694	0.20975	0.29003
50	0.01000	0.03248	0.08298	0.15002	0.20975
75	0.00806	0.02646	0.06782	0.12294	0.17257
100	0.00707	0.02291	0.05878	0.10668	0.15002

Table 8: Max Triangular Extreme Interval Values when a = 0, b = 1, and c = 0.50

n	**α = 0.01**	**α = 0.10**	**α = 0.50**	**α = 0.90**	**α = 0.99**
1	0.07071	0.22361	0.50000	0.77639	0.92929
5	0.44616	0.57044	0.74559	0.89790	0.96830
10	0.57044	0.67932	0.81701	0.92761	0.97764
25	0.70997	0.79025	0.88306	0.95412	0.98586
50	0.79025	0.84998	0.91702	0.96752	0.99000
75	0.82743	0.87706	0.93218	0.97354	0.99194
100	0.84998	0.89332	0.94122	0.97709	0.99293

Table 9: Min Triangular Extreme Interval Values when a = 0, b = 1, and c = 0.50 (Jance and Thomopoulos)

n	α = 0.01	α = 0.10	α = 0.50	α = 0.90	α = 0.99
1	0.07071	0.22361	0.50000	0.77639	0.92929
5	0.03169	0.10211	0.25441	0.42956	0.55385
10	0.02241	0.07239	0.18298	0.32068	0.42956
25	0.01418	0.04586	0.11693	0.20975	0.29003
50	0.01002	0.03244	0.08297	0.15001	0.20975
75	0.00819	0.02649	0.06782	0.12295	0.17256
100	0.00709	0.02295	0.05877	0.10668	0.15001

Table 10: Max Triangular Extreme Interval Values when a = 0, b = 1, and c = 0.50 (Jance and Thomopoulos)

n	α = 0.01	α = 0.10	α = 0.50	α = 0.90	α = 0.99
1	0.07071	0.22361	0.50000	0.77639	0.92929
5	0.44615	0.57044	0.74559	0.89789	0.96831
10	0.57044	0.67932	0.81702	0.92761	0.97759
25	0.70997	0.79025	0.88307	0.95414	0.98582
50	0.79025	0.84999	0.91703	0.96756	0.98998
75	0.82744	0.87705	0.93218	0.97351	0.99181
100	0.84999	0.89332	0.94123	0.97705	0.99291

CONCLUSIONS AND RECOMMENDATIONS

This paper shows how the min and max extreme interval values for the exponential and triangular distributions can be found from the min and max uniform extreme interval values. Tables displaying the min and max extreme interval values for these distributions are presented for different observation sizes n and probabilities α. One could use the min and max uniform extreme interval values to find the min and max extreme interval values for other types of continuous probability distributions (e.g. normal, lognormal) in the future.

AUTHOR INFORMATION

Marsha L. Jance received her Ph.D. in Management Science from Illinois Institute of Technology Stuart School of Business in Chicago, Illinois. She is an Assistant Professor of Business Administration at Indiana University East in Richmond, Indiana. E-mail: mjance@iue.edu

Nick T. Thomopoulos is a professor at the Stuart School of Business at Illinois Institute of Technology. Nick has degrees in business, mathematics, and industrial engineering. He is the author of four books: *Quantitative Methods Along the Supply Chain*, Atlantic Books, 2011, *Assembly Line Systems*, Hayden Books, 1973, *Applied Forecasting Methods*, Prentice Hall, 1980, and *Strategic Inventory Management and Planning*, Hitchcock Publishing Co., 1990. E-mail: thomop@stuart.iit.edu

REFERENCES

1. Hines, William W., Douglas C. Montgomery, David M. Goldsman, and Connie M. Borror. *Probability and Statistics in Engineering Fourth Edition*. John Wiley & Sons, Inc., 2003.
2. Jance, Marsha L. Calculating Min and Max Extreme Interval Values for Various Distributions. Doctoral dissertation. Chicago, IL: Illinois Institute of Technology Stuart School of Business, 2007.
3. Jance, Marsha L. and Nick T. Thomopoulos. "Min and Max Triangular Extreme Interval Values and Statistics." *Journal of Business and Economics Research* 8.9 (2010): 139-146.
4. Law, Averill M., and W. David Kelton. *Simulation Modeling and Analysis Third Edition*. McGraw-Hill, 2000.

NOTES

