A Stochastic Present Value Model In Selecting Risk Management Processes

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Abstract

The analysis of stochastic present value models provides one of the more powerful techniques applicable to financial decision making under conditions of uncertainty. The paper introduces a present value model for a random payment decomposed into a random sum of continuous positive independent and identically distributed random variables, under a random timing represented as the minimum of a random number of continuous positive and identically distributed random variables. Properties of the corresponding distribution function are established. Moreover, the paper provides applications of the model in selecting risk management processes for a system consisting of a random number of components, each component having a random failure time.

1. Introduction

Any corporation implies self-regulation and permanent choices, not only in economic, financial and technical matters, but also at the social level. Not only the capital of the corporation has to be valorized and protected, but also the human components. Specific risks correspond to all levels of the corporation generating one or several types of exposure. It is important to note that few corporations try to consider risks as an integral part of the entire system. And that explains why risk management philosophy is not always successful. To be an integral part of the corporation, risk management must be global and systemic.

Risk management is essentially a prospective activity, concentrating on risks threatening a corporation and on measures taken now or in the future that can reduce the negative consequences of these risks. Modern risk management is moving from handling strictly pure risks to a wider concern for dynamic risks, taking into consideration the interplay of all risks in corporate operations. It must be recognized that risk management is still in a development stage. As a developing discipline, risk management is still an art rather a science. While it has roots in security management, in the growth of insurance industry, and in the industry's response to needs, it will be taken new forms in the future.

As currently defined, the elements of risk management include risk identification, risk measurement, risk treatment and administrative techniques to carry out the risk management

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process more effectively. The work presented here is related to selecting risk treatment processes. Once the risk has been identified and measured, the various processes of risk treatment should be considered and a decision made with respect to the best combination of processes to be applied in attacking the problem. These processes include control and financing of risk. In selecting the proper process the risk manager must establish the costs and other consequences of applying its process or combination of processes. The risk manager must also consider the present financial position of the corporation, its overall policy with reference to risk management, its specific objectives and the effectiveness of different risk management processes in protecting against the negative consequences of risks. To put a corporation's risk management efforts to best use a risk manager must be able to forecast the risks a corporation is likely to go through in a proactive manner.

The time value of money makes necessary the application of present value models in selecting risk treatment processes for a proactive risk management program. Furthermore, the random character of severity, timing and other components of risk requires these models to be stochastic. With stochastic present value models, the risk manager can obtain valuable information concerning selection and implementation of risk treatment processes and techniques. Once such models are available, it is possible to intelligently evaluate the issues and alternatives and chart courses of action for a proactive risk management program. Many stochastic present value models have been readily available to corporate risk managers for making risk management decisions. These models are being used to improve not only the decisions that must be made, but also in the presentations to top management on critical issues. As risk managers begin to use stochastic present value models more skillfully they will become a more important part of the long-range planning team.

The stochastic treatment of present value models takes advantage of the results of probability distribution theory and aims to provide management with more detailed and more realistic criteria upon which to base risk management decision making. Since in general any analytical determination of the distribution function of a stochastic present value model is very difficult, it follows that the establishment of some properties of this distribution function can be proved very important within the risk management decision making process. Recently, several extensions to the study of stochastic present value models are due to Artikis and Voudouri (1987), Voudouri (1989), Artikis and Malliaris (1990), Artikis and Jerwood (1991), Artikis, Jerwood and Voudouri (1992), Artikis, Voudouri and Jerwood (1993), Artikis (1994), and Jerwood and Moshakis (1995).

These contributions concentrate specifically on the conditions under which the properties of α-unimodality and ν-unimodality will be introduced into the distribution functions of some stochastic present value models. These properties are extensions of the property of ordinary unimodality which is a welcome feature on stochastic modeling operations. From a practical point of view the above articles provide new stochastic financial derivations for the classes of α-unimodal and ν-unimodal distributions which are very important in the theory of compound distributions. The work presented here extends the results of the above contributions.

The purpose of the paper is to introduce a new stochastic present value model. Properties and applications, in selecting risk management processes, of this model are given. The main part of the paper consists of five sections. Section two is devoted to the introduction of the stochastic present value model. Applications of the model in portfolio and risk management are provided in section three. Section four is devoted to the derivation of the distribution function of the model. Concluding remarks and suggestions for further research in the area of stochastic discounting modeling operations are given in sections five and six respectively.

2. The Model

The purpose of this section is to introduce a new stochastic present value model. Let

$$\{X_k, \quad k = 1,2,\ldots,\}$$

be a sequence of continuous positive independent and identically distributed random variables. Let K be a discrete random variable taking values in $\{0,1,2,\ldots,\}$ and independent of the random variables $X_k, \quad k = 1,2,\ldots,$ and consider the random sum

$$Y = \begin{cases} 0 & , \quad K = 0 \\ X_1 + X_2 + \ldots + X_K & , \quad K \geq 1. \end{cases}$$

Moreover, let

$$\{S_n, \quad n = 1,2,\ldots,\}$$

be a sequence of continuous positive independent and identically distributed random variables and N be a discrete random variable taking values in $\{1,2,\ldots,\}$ and independent of the random variables $S_n, \quad n = 1,2,\ldots,$. We set

$$T = min\{S_1, S_2, \ldots, S_N\}$$

and assume that the random variables Y, T are independent.

The purpose of the paper is to establish properties and applications of the stochastic model

$$V = Ye^{-rT} , \qquad (2.1)$$

where $r > 0$. If r is the nominal compound interest rate then the above stochastic model denotes the present value V of a random cash flow Y, which is the random sum of continuous positive independent and identically distributed random variables, under a random timing T, which is the minimum of a random number of continuous positive independent and identically distributed random variables.

The notion of the sum of a random number of independent and identically distributed random variables and the notion of the minimum of a random number of independent and identically distributed random variables are very important in modeling operations. The main theoretical contribution of this paper is the use of these notions in a stochastic present value model.

From a practical point of view the presence of the random sum Y and the minimum T in the stochastic present value model (2.1) extends the applicability of this model in financial decision making under conditions of uncertainty.

The distribution function of V is very complicated, but the possibility of applying numerical methods and approximations to this distribution function facilitates the use of the above present value model in practical situations. Furthermore, Monte Carlo simulation methods provide a viable means to verify the implications of the stochastic present value model provided by this paper.

3. Applications Of The Model

During the last twenty years risk management expanded rapidly on a global, systemic and proactive basis. Risk identification, risk measurement and risk treatment have become far more sophisticated, using new quantitative models. The contribution of the quantitative models to the development of risk management as an organizational discipline has been proved very important. The stochastic present value models constitute a significant part of the quantitative models applied to risk management problems. Significant research topics in the area of stochastic modeling can be considered the development, study and applications of stochastic present value models in selecting processes for evaluation and treatment of risk.

Systems consisting of a random number

of components, each component having a random life-time, are very common in many applied fields and the completion of their risk profile is a very crucial element in developing and implementing risk management programs for these systems, Tijms (1988). Hence a stochastic present value model whose timing is the minimum of a random number of random life-times provides valuable information concerning the risk profile of such systems. The risk profile is an attempt to state the unique risk characteristics and unique risk situation of a particular system at a particular time. When this profile is completed broad risk objectives usually in terms of risk control and risk financing can be determined. Before preparing detailed plans for achieving these overall objectives, an audit of existing risk management, the practical measures by which management actually handle the risk, will show the practicality of the overall objectives. In this section, we provide an application of the above stochastic present value model in portfolio management.

At time 0 we consider a portfolio of N investments, where N is a discrete random variable taking values in $\{1,2,\ldots,\}$. We suppose that N is independent of the sequence of continuous positive independent and identically distributed random variables $\{S_n,\ \ n=1,2,\ldots,\}$, where S_n denotes the duration of the n-th investment. The random time

$$T = min\{S_1, S_2, \ldots, S_N\}$$

is very crucial in considering selection and implementation of risk management processes which will improve the performance of the portfolio. More precisely the introduction of T in stochastic discounting operations provides management with valuable information upon which to base analysis and control of risk.

We suppose that the investment which ends at time T is replaced immediately by K new investments, where K is a discrete random variable taking values in $\{0,1,2,\ldots,\}$. Moreover, we suppose that K is independent of the sequence of continuous positive independent and identically distributed random variables $\{X_k,\ \ k=1,2,\ldots,\}$, where X_k denotes the cost of the k-th new investment.

The random variable

$$Y = \begin{cases} 0 & , \quad K = 0 \\ X_1 + X_2 + \ldots + X_K & , \quad K \geq 1 \end{cases}$$

denotes the cost of the K new investments which will replace the investment which ends at time T. The stochastic present value model

$$V = Ye^{-rT}$$

is important in the study of the future performance of the above portfolio of investments.

The above stochastic present value model is also important for studying the evolution of any economic system consisting of a random number of components, each component having a random failure-time. For example, the subsidiaries of a firm at a given time can be considered as a system of this type.

The random variable T denotes the time the first failure occurs. A stochastic present value model with timing T provides valuable information for selecting a risk treatment technique at time 0. The purpose of such a technique will be the avoidance, reduction or financing of the negative consequences due to the first failure. The incorporation of the random variable T in the stochastic present value model proposed by the paper implies that this model can be a useful tool for treatment of risk. More precisely, the proposed model provides management with the required information to evaluate the advantages of using a random number of components to replace the component which fails at time T as a tool of a proactive risk management program.

4. Distribution Of The Model

This section is devoted to the distribution of the present value V in (2.1). Since the distribution function $F_V(v)$ of V is very complicated it is useful to derive the corresponding characteristic function $\phi_V(u)$.

Let $F_S(s)$ be the distribution function of the random variable S_n, $n = 1,2,\ldots,$ and $P_N(z)$ be the probability generating function of the random variable N. Consider the random variable

$$W = e^{-rT} \ , \ r > 0 .$$

It can be shown that the distribution function of W is given by

$$F_W(w) = P_N\left(1 - F_S\left(-\frac{1}{r} \log w\right)\right) , \ 0 \le w \le 1 . \tag{4.1}$$

Let $\phi_X(u)$ be the characteristic function of the random variable X_k, $k = 1,2,\ldots,$ and $P_K(z)$ be the probability generating function of the random variable K. It is easily demonstrated that

$$\phi_Y(u) = P_K(\phi_X(u)) \tag{4.2}$$

is the characteristic function of Y, Feller (1966). From the assumptions of the present value model (2.1) it follows that the random variables W, Y are independent. From (4.1) and (4.2) it follows that the integral

$$\phi_V(u) = \int_0^1 P_K(\phi_X(uw)) dF_W(w) \tag{4.3}$$

defines the characteristic function of V. The use of the characteristic function $\phi_V(u)$ provides a method for calculating the corresponding distribution function $F_V(v)$. The solution makes use of Fourier inversion of the characteristic function $\phi_V(u)$. By first obtaining and then inverting the characteristic function $\phi_V(u)$, one can obtain the distribution function $F_V(v)$. The use of a computational algorithm known as the Fast Fourier Transform makes this process more manageable by greatly reducing computational time, Paulson and Dixit (1989). Fourier analysis is extremely powerful and could eventually revolutionize the study of distribution functions of complicated stochastic present value models. Acceptance of the method has been slowed by the relatively advanced nature of the mathematics it entails and by the absence of computer programs that have been adapted for convenient use in financial applications. As these barriers are overcome, use of the method to analyze problems in finance will become more common.

In general the characteristic function $\phi_V(u)$ is very complicated however in the particular case when $P[N = n] = 1$ and the random variables $S_1, S_2, \ldots, S_n$ are exponentially distributed with parameter λ it follows that the random variable T is exponentially distributed with parameter $n\lambda$ and the distribution function of the random variable W follows the standard power distribution with parameter $\alpha = n\lambda / r$. In this case the characteristic function of the random variable V is of the form

$$\phi_V(u) = \alpha \int_0^1 P_K(\phi_X(uw)) w^{\alpha - 1} dw . \tag{4.4}$$

Characteristic functions of the form (4.4) belong to α-unimodal distribution functions, Olshen and Savage (1970). Since the exponential distribution is the most commonly encountered timing distribution and the minimum of a given number of independent exponentially distributed random variables is also exponentially distributed it follows that the class of α-unimodal distribution functions is very important in the study and applications of the above particular case of the present value model (2.1). The establishment of some properties of the class of α-unimodal distribution functions can be proved invaluable within the financial decision making process. Unimodality, besides being a well known property in probability and statistics, is a property with practical and theoretical interest. In a practical situation, unimodality helps to obtain better statistical inferences. The existence of a unique mode is clearly essential, since the presence of multimodality will introduce a degree of localized ambiguity into the decision making process. Unimodality generally is a welcome feature from the point of view of constructing unambiguous estimation procedures. Moreover, unimodal distributions are invariably more realistic in the eyes of economists than the extremal distributions resulting when one does not require unimodality. The conditions which allow unimodality to be introduced into the distribution of a stochastic present value model need to be identified and such conditions need to be encouraged in modeling operations. When $0 < \alpha \leq 1$ the α-unimodal distributions have unique mode at the point 0.

5. Concluding Remarks

The stochastic present value model in (2.1), with the random payment Y decomposed into a random sum of continuous positive independent and identically distributed random variables and the random timing T represented as the minimum of a random number of continuous positive independent and identically distributed random variables, provides management with valuable information concerning the risks of an economic system consisting of a random number of components. This means that the model (2.1) can be of some practical importance in selecting and implementing risk management processes which will improve the performance of the system.

The distribution function $F_V(v)$ corresponding to the stochastic present value model in (2.1) is very complicated and hence any explicit analytical determination of $F_V(v)$ is extremely difficult. However, the possibility of applying numerical methods and approximations to $F_V(v)$ facilitates the use of the proposed stochastic present value model in selecting and implementing risk management processes. Moreover, Monte Carlo simulation techniques provide a viable means, in some practical way, to verify the consequences in risk management applications of the stochastic present value model provided by the paper.

A powerful method for the calculation of the distribution function $F_V(v)$ of the stochastic present value model in (2.1) is provided by using the characteristic function $\phi_V(u)$ which corresponds to $F_V(v)$. Moreover, the use of the Fast Fourier Transform makes the inversion of the characteristic function $\phi_V(u)$ more manageable by greatly reducing computational time.

6. Suggestions For Further Research

Research activities in the area of stochastic present value models applied to evaluation and selection of risk management processes are motivated by the following reasons. First, the development, study and applications of stochastic present value models incorporate ideas from probability theory, statistics, economics, decision theory and informatics. Second, the constituent disciplines of risk management include probability theory, statistics, economics, operations research, systems theory, decision theory, psychology and behavioral science.

It seems that further research should be carried out on the properties and the financial applications of a particular case of the stochastic present value model

$$V = Ye^{-rT} , \qquad (6.1)$$

where the random payment Y and the random timing T are continuous positive and independent random variables, and the nominal continuous compound interest rate r is constant.

An interesting particular case of the stochastic present value model in (6.1) arises when the random payment Y has the form

$$Y = \begin{cases} 0 & , \quad K = 0 \\ X_1 + X_2 + \ldots + X_K & , \quad K \geq 1, \end{cases}$$

with K a discrete random variable which follows the binomial distribution

$$P[K = k] = \binom{m}{k} p^k (1-p)^{m-k},$$

where $0 < p < 1$ and $k = 0,1,2,\ldots,m$.

Moreover, in this particular case the random timing T will have the form

$$T = min\{S_1, S_2, \ldots, S_N\},$$

with N a discrete random variable which follows the geometric distribution

$$P[N = n] = \vartheta (1 - \vartheta)^{n-1},$$

where $0 < \vartheta < 1$ and $n = 1,2,\ldots,$.

The study of the above particular case of the stochastic present value model in (6.1) is very interesting from a theoretical and a practical point of view. From a theoretical point of view the establishment of unimodality, stability, infinite divisibility and some other properties for the distribution function of the above particular case of the model in (6.1) is important since, in general, any explicit analytical determination of this distribution function is extremely difficult.

Furthermore, from a practical point of view it seems that the above particular case of the model in (6.1) is of some importance in financial analysis and decision making under conditions of uncertainty, since the binomial distribution and the geometric distribution have interesting practical applications in stochastic modeling of many financial processes and events. 🕮

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