OPTIMAL EXPENDITURE POLICY UNDER VARYING LEVELS OF CAPITAL ALLOCATIONS AND COSTS

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Abstract

The paper addresses the problem of allocation of resources (budget) faced by various administrators in public and private sectors. These administrators would like to maintain the level of service that they provide but find it difficult in view of varying levels of costs and budgets. Using some basic elements of Financial Theory, this paper helps solve this problem under certain market conditions.

1. INTRODUCTION:

This paper is concerned with a capital allocation problem faced typically by certain service providers such as hospital administrators, employee health care insurance and fringe benefit administrators, acquisition librarians, etc. These administrators would like to improve the level of service every year but capital constraints may not allow that to happen. In the event of limited capital budget, the administrators would at least like to maintain the current level of service in the near future. However, even the maintenance of the current level of service may not be easy if capital allocation and costs of the services were to vary from year to year. For example, the librarian may have a yearly budget allocation with which to purchase professional journals. If price and budgetary fluctuations occur, it is not desirable to discontinue journals for a year or two, nor is it of benefit to purchase additional materials for only a few years. The greatest benefit can be derived from a constant set of purchases, which aids the researcher by providing continuity of reference documents.

The problem facing the administrators, such as librarians, etc. becomes complex. The administrators

cannot simply maintain their dollar expenditure constant but they have to maintain the level of service constant in the light of changing levels of unit costs and budget allocations. How to achieve the objective of maximum constant service under the constraints imposed by varying levels of changes in costs and budgets is the subject matter of this paper.

The solutions discussed in this paper rest on one important assumption. It is assumed that the capital market are accessible to the service providers. This assumption will allow the service providers to enlarge their choice set: they can invest the unspent budget for future use or borrow in anticipation of future budget increases in order to smoothen the expenditure pattern over a time period.(1)

The procedure employed in this paper consists of equating present values of budget (inflow) and expenditure (outflow) so as to maximize the (constant) level of service in the face of various scenarios. The scenarios (states of nature) are distinguished by various combinations of expenditure and budget behavior over time, which are described below:

	BUDGET:	rising $b > 0$	falling $b < 0$	constant $b = 0$
UNIT COST:				
rising ($c > 0$)		+, +	+, -	+, 0
falling ($c < 0$)		-, +	-, -	-, 0
Constant ($c = 0$)		0, +	0, -	0, 0

In the above table, (b) indicates the (annual) rate of change in budgetary allocation. The budgetary allocation may be rising ($b > 0$), falling ($b < 0$) or may remain constant ($b = 0$). The service administrator is expected to spend this budget by purchasing certain goods and/or services. The purchases are influenced by cost per unit (c); which may also be either rising ($c > 0$), or falling ($c < 0$) or remain constant ($c = 0$). The signs (+, -) in the table indicate the direction of change per period of cost and budget allocations.(2) Thus, the table describes a total of nine states of nature which can be classified into two types. The first type of cases are those where either cost or budget or both are constant (i.e., $c = 0$ or $b = 0$ or $c = b = 0$); the second type of situations are those where neither cost (c) nor budget (b) remain constant.

Before proceeding further it might be useful to define the variables used in the rest of the paper:

- A: The total expenditure per period, equals the number of units consumed multiplied by the cost per unit (c). A (0) indicates level of expenditure at time 0.
- B: The budget allocations. B (0) indicates budget allocation at time 0.
- C: The cost per unit. C (0) indicates cost per unit at time 0.
- b: The annual rate of growth of budget allocation (B), more specifically it is dB/dt .
- c: The rate of change in cost per unit

(C), more specifically it is dC/dt .

n: The number of periods in the horizon for a state of nature.

PV: The present value of a stream of budget (B) or expenditure (A).

r: The rate of interest.

2. CONSTANT COST AND/OR BUDGET SITUATIONS:

A total of five cases are discussed under the assumption that cost per unit is expected to remain constant but budgetary allocation would either increase or fall (2 cases), budget is expected to remain constant but cost per unit is expected to either increase or fall (2 cases), and finally a trivial case where both cost and budget would remain constant.

Case 2.1: Constant Cost But Rising Budget ($c = 0, b > 0$)

The cost per unit is assumed to remain constant, but the budgetary allocation is expected to increase every year at a rate of (b). Assume for the moment that only the budgeted amount is spent in the first year. There remains a surplus in the second year because the budget has grown and the costs have not. It is, therefore, possible to increase the current level of service by borrowing in anticipation of future budgetary increases. The borrowed funds will be paid back in the later years when the budget exceeds the spending levels.

The annual expenditure (A) is equal to the cost per unit (C) multiplied by number of units purchased. The cost per unit is assumed to remain unchanged and the objective of constant services calls for number of units to be constant. Thus, the annual expenditure (A) is to be kept unchanged and can be expressed as an annuity. The budgetary allocation (B) expected to increase at an annual rate of (b), can be expressed by a geometric sequence. The solution, therefore, requires that the present value of future annual expenditure (annuity) equal the present value of future annual budgetary allocations (geometric series).

The expression for the present value (PV) of an annuity is developed below, where annual expenditure is denoted by (A), and the rate of interest is assumed to be (r):

$$\begin{aligned}
 PV &= \frac{A}{(1+r)} + \frac{A}{(1+r)^2} + \frac{A}{(1+r)^3} + \dots + \frac{A}{(1+r)^n} \\
 &= \frac{A}{(1+r)} \sum_{k=0}^{n-1} \frac{1}{(1+r)^k} \\
 &= A \{1 - [1/(1+r)]^n\} / r \quad (1)
 \end{aligned}$$

The present value (PV) of a geometric sequence can also be expressed. Let the initial budget be equal to the amount (B) which is expected to increase at an annual rate of (b).

$$\begin{aligned}
 PV &= \frac{B}{(1+r)} + \frac{B(1+b)}{(1+r)^2} + \frac{B(1+b)^2}{(1+r)^3} + \dots + \frac{B(1+b)^{n-1}}{(1+r)^n} \\
 &= \frac{B}{(1+r)} \sum_{k=0}^{n-1} \frac{(1+b)^k}{(1+r)^k} \quad (2)
 \end{aligned}$$

Equating equations (1) and (2) will help us solve for (A), the annual expenditure. However, equation (2), which gives the present value of a finite geometric series, describes a general case but it can be simplified to incorporate situations

where $b = r$, and $b \neq r$.

In the case $b = r$, equation (2) becomes:

$$PV = \frac{nB}{(1+r)} \quad (2.a)$$

In the case $b \neq r$ the finite geometric series in (2) can be modified to:

$$PV = [B/(r-b)] \{1 - [(1+b)/(1+r)]^n\} \quad (2.b)$$

The situation where costs remain constant but the budget is rising can be solved by equating the R.H.S. of equations (1) and either (2.a) or (2.b) depending upon the relationship between the budget growth rate (b) and the rate of interest (r):

Assuming $b=r$, the R.H.S. of equation (1) can be set equal to the R.H.S. of equation (2.b):

$$A \{1 - [1/(1+r)]^n\}/r = B/(r-b) \{1 - [(1+b)/(1+r)]^n\}$$

Therefore:

$$A = \frac{Br}{(r-b)} \cdot \frac{\{1 - [(1+b)/(1+r)]^n\}}{\{1 - [1/(1+r)]^n\}} \quad (3)$$

If $b=r$, then the R.H.S. of equation (1) is set equal to the R.H.S. of equation (2.a):

$$A = \frac{nB}{(1+r)} \cdot \frac{r}{\{1 - [1/(1+r)]^n\}} \quad (4)$$

The solutions presented in (3) and (4) indicate that by spending the amount equal to (A) every year the service administrator would be able to purchase the number of units equal to A/C . In the initial year, expenditure (A) will be greater than the budget (B), but this would be feasible because the present value of the expenses (A) equals the present value of the budgetary allocation (B).

These and other solutions that follow are illustrated also with a numerical example which is provided in the Appendix.

Now the behavior is investigated as the number of periods increases without bound. Three cases are discussed: the interest rate (r) is assumed to be greater than the budget growth rate (b), less than the growth rate, and equal to the growth rate.

From equation (3) it can be seen that $b > r$ implies that the value of (A) diverges as n increases without bound. In the same equation, if $b < r$ it is seen that A approaches $Br/(r-b)$ when n increases without bound. Finally, if $b = r$ it is seen that A diverges as n increases in equation (4).

Case 2.2: Constant Cost But Falling Budget ($c=0$, $b<0$)

The budget is assumed to decline at an annual rate of (b). In this case the entire budgeted amount cannot be spent in the first year because that would cause a reduction of level of service in future years due to a declining budget. Thus, part of the current budget should be invested and withdrawn later when the budget is expected to fall, so that constant service can be maintained.

As in the previous case, equation (3) can be applied to find the solution. The case of the interest rate (r) being equal to the budget growth rate (b) need not be considered because the growth rate is negative and the interest rate is positive. Equation (3) is reproduced below, but note that b is negative in this case.

$$A = \frac{Br}{(r-b)} \cdot \frac{\{1 - [(1+b)/(1+r)]^n\}}{\{1 - [1/(1+r)]^n\}}$$

Since both $(1+b)/(1+r)$ and $1/(1+r)$ are less than 1, it is assured that their powers approach 0 as n increases. This results in A approaching $Br/(r-b)$.

Case 2.3: Rising Cost But Constant Budget ($c>0$, $b=0$)

This is quite parallel to case 2.2 because rising cost with constant budget can be interpreted as constant cost and falling budget. Therefore, as in case 2.2, part of the budget must be invested in the early years and withdrawn later to cover the rising costs.

If cost per unit is assumed to rise every year, the annual expenditure (A) must also rise in order to maintain constant service. Thus, the annual expenditures can no longer be expressed as a series of an annuity. For this reason, only the initial level of spending, denoted as $A(0)$, is derived in equation (5) below. Once the initial year expenditures is known, the number of units to be purchased is easily determined by dividing the initial expenditure $A(0)$ by cost per unit in the initial year $C(0)$.⁽³⁾ For subsequent years, the number of units purchased is kept constant, but in the event of rising cost the annual expenditure (A) will keep on rising.

The general expression for $A(0)$, when the rate of change in price per unit (c) is equal to the interest rate (r), is derived by using equations (1) and (2.a).

$$A(0) = \frac{B(1+r)}{nr} \{1 - [1/(1+r)]^n\} \quad (5)$$

If c and r are not equal, equations (1) and (2.b) can be used:

$$A(0) = \frac{B(r-c)}{r} \cdot \frac{\{1 - [1/(1+r)]^n\}}{\{1 - [(1+c)/(1+r)]^n\}} \quad (5.a)$$

When n increases, $A(0)$ approaches 0 in equation (5). If c exceeds r , A again approaches 0, but if c is less than r the second term in the R.H.S. of (5.a) approaches 1 and A approaches $B(r-c)/r$.

Case 2.4: Falling Cost But Constant Budget ($c < 0$, $b = 0$)

In this case, it is possible to spend more than the initial allocation, as falling prices in later years would provide a source of income. As in Case 2.2, equations (1) and (2.b) can be used to determine the initial level of spending.

The general formula for $A(0)$ is:

$$A(0) = \frac{B(r-c)}{r} \cdot \frac{\{1 - [1/(1+r)]^n\}}{\{1 - [(1+c)/(1+r)]^n\}}$$

As n increases, the second terms on the R.H.S. approach 1, so A approaches $B(r-c)/r$. (4)

Case 2.5: Constant Cost and Constant Budget ($c = b = 0$)

With both budget and costs constant, the budgeted amount is spent each year. No further study is made of this trivial case.

3. RISING COST AND RISING BUDGET ($c > 0$, $b > 0$):

This case is considered separately because there are three subcases which arise because of the different relationships between the rates of interest (r) and the budget growth (b).

Case 3.1a: Both Cost and Budget Rise Equally ($c = b$)

This case is similar to ($c = b = 0$). The budgeted amount can be spent in the first year, and then the level of service remains constant because budget and costs remain equal through a constant rate of growth for each.

Case 3.1b: The Budget Rises Faster Than Cost ($b > c$)

More than the allocated budget is spent in the early years by borrowing against the surplus of later years.

The general expression for $A(0)$ is obtained by using equation (2.b) to obtain

the present value of both the budgeted amounts and the expenditures.

The present value of the budgeted amounts is:

$$PV = B/(r-b) \{1 - [(1+b)/(1+r)]^n\}$$

The present value of the expenditures is:

$$PV = A(0)/(r-c) \{1 - [(1+c)/(1+r)]^n\}$$

Equating the present values gives:

$$A(0) = \frac{B(r-c)}{(r-b)} \cdot \frac{\{1 - [(1+b)/(1+r)]^n\}}{\{1 - [(1+c)/(1+r)]^n\}} \quad (6)$$

As n gets large there are five cases to consider, for the interest rate can be greater than b , equal to b , between b and c , equal to c , or less than c .

In the first case ($r > b > c$), $A(0)$ approaches $B(r-c)/(r-b)$, since the second terms on the R.H.S. approach 1.

In the second case ($r = b > c$), the expression for A becomes

$$A = \frac{B(b-c)}{(1+b)} \cdot \frac{n}{\{1 - [(1+c)/(1+r)]^n\}} \quad (7)$$

Clearly A diverges as n increases.

A diverges in the third case ($b > r > c$) because the quotient that is raised to the power of n in the numerator is greater than 1, while that in the denominator is less than 1.

The fourth case ($b > r = c$) involves again another expression for A , namely:

$$A(0) = \frac{B(1+r)}{(r-b)} \cdot \frac{\{1 - [(1+b)/(1+r)]^n\}}{n} \quad (8)$$

Because the numerator of the second term on the R.H.S. dominates the denominator, this also gives divergence for A .

Last, the case ($b > c > r$) is considered where the quantities $(1+b)/(1+r)$ and $(1+c)/(1+r)$ both diverge. By L'Hopital's Rule, the second term of the R.H.S.

of equation (6) behaves like $(1+b)/(1+c)$. Since this quotient exceeds 1, ratio of the second terms of the R.H.S. diverges, as does A.

Case 3.1c: Cost Rises Faster Than The Budget ($c > b$)

The entire budgeted amount cannot be spent in the earlier years because are expected to rise faster in the future. The excess must be invested and then withdrawn in later years.

The general solution is the same as in equation (6) for Case 3.1b. However, five cases again must be considered. The interest rate can exceed both c and b, equal c, lie between c and b, equal b, or be less than both c and b.

As previously, in the first situation ($r > c > b$) A converges to $B(r-c)/(r-b)$ because as n increases both $(1+b)/(1+r)$ and $(1+c)/(1+r)$ are less than unity and approach 0 when raised to the increasing powers.

The solution for the second case ($r = c > b$) is given in equation (8), so A approaches 0 as n increases.

The third case ($c > r > b$) results in A approaching 0 because the numerator decreases and denominator increases (in absolute value) in equation (6) as n increases.

In the fourth case ($c > r = b$) the power term dominates n as before, causing the quotient to approach 0 in equation (7).

Using the same reasoning as in the fifth part of Case 1b, the ratio $(1+b)/(1+c)$ raised to larger powers approaches 0, causing A to approach 0 also in equation (6) for the final case ($c > b > r$).

4. FALLING COST AND RISING BUDGET ($c < 0$, $b > 0$):

In this case there are two reasons for initially spending more than the budgeted amount. The increase in budget and reduction in the cost of service will both give a source of repayment for borrowing to be made in the early years.

Using equation (2.b) to obtain expressions for the present value of both the budgets and expenditures and then equating them given an expression for the initial level of spending when (b) does not equal (r):

$$A(0) = \frac{B(r-c)}{(r-b)} \cdot \frac{\{1 - [(1+b)/(1+r)]^n\}}{\{1 - [(1+c)/(1+r)]^n\}}$$

If (b) is greater than (r), then A diverges as n increases, but if (b) is less than (r), A approaches $B(r-c)/(r-b)$.

CONCLUDING REMARKS

In this paper, some solutions were provided to help resolve certain problems that are likely to be faced by administrators responsible for maintaining constant service. The assumptions and methodology used in this paper are similar to those found concerning the consumption investment decisions so often discussed in the context of the theory of finance [see for example Fama and Miller (1972), Copeland and Weston (1983)]. To the extent these assumptions can be met the solutions provided in this paper should be of great help to those responsible for providing constant level of service for their clientele such as librarians, employee benefit and health care administrators, and many others. In the event that some of the assumptions are not met, the solutions of this paper can be altered to suit individual situations.

FOOTNOTES

1. This assumption is made very often in the literature, while describing the consumption-investment decision in the presence of Capital markets. For example see: Fama & Miller (1972) or Copeland and Weston (1983).
2. How one determines the rate of growth of cost and budget is a separate issue. However we find comfort in the fact that similar assumptions are made in the literature regarding the rate of growth of dividends in the context of common stock valuation. For example see Brealey and Myers (1984) p. 48-50.
3. The discussion here assumes that cost per unit is constant for all the purchases made. This assumption is valid only if all the purchases happen to be homogeneous such as employee health care insurance. However, purchases of different journals made by a librarian are not likely to be homogeneous and, therefore, per journal cost is likely to be different. But as long as the first year expenditure, $A(0)$, is known the librarian can decide how many and which journals to buy in the first year and continue to purchase the same journals in the future, provided the average cost of the journals keeps rising at the rate of (c) .
4. The limit which A approaches does not depend on the relative size of (c) compared to (r) because the former is a rate of decline (a negative rate) and the latter is assumed to be positive.

REFERENCES

- Brealey, R. and S. Myers, *Principles of Corporate Finance*, 2ed., McGraw Hill, 1984.
- Copeland, T. E. and J. F. Weston, *Financial Theory and Corporate Policy*, Addison Wesley, 1983.
- Fama, E. F., and M. H. Miller, *The Theory of Finance*, Holt, Rinehart and Winston, New York, 1972.

When (b) and (r) have the same value, equations (2.a) and (2.b) give the value for A(0):

$$A(0) = \frac{B(b-c)}{(1+b)} \cdot \frac{n}{\{1 - [(1+c)/(1+r)]^n\}}$$

This also results in A(0) diverging as n increases.

5. VARIABLE COST AND FALLING BUDGET (c>0, b<0):

Two cases are considered where the budget falls at a constant rate. The costs must either rise or fall at a given annual rate.

Case 5.1: The Cost Rises And Budget Falls (c>0, b<0)

This case again suggests that the entire budget should not be spent in the early years because the later years will support a lower level of spending.

The behavior as n gets large is studied in three cases: r exceeds c, r equals c, or r is less than c.

The first case again uses equation (6), as the first case of Case 3 part c, so it is seen that A(0) approaches B(r-c)/(r-b).

Cases two and three are likewise identical in formula to the corresponding parts of Case 3 part c, so A will approach 0, being assured that b is less than r.

Case 5.2: Both Cost and Budget Fall (c<0, b<0)

Should these occur at the same rate, then it is clear that the budgeted amount can be spent each year and a constant level of service can be maintained. If costs fall more rapidly, borrowing should be done initially: if the budget falls more rapidly, investment should be done initially.

The general equation used to determine A is

$$A(0) = \frac{B(r-c)}{(r-b)} \cdot \frac{\{1 - [(1+b)/(1+r)]^n\}}{\{1 - [(1+c)/(1+r)]^n\}}$$

This converges in all cases to B(r-c)/(r-b).

APPENDIX Numerical Illustrations

Case	BUDGET		COST		(r)	n	A
	rising (b>0)	falling (b<0)	rising (c>0)	falling (c<0)			
2.1(0,+)	10%				15%	5	\$119*
	10%				15%	10	\$143*
	10%				15%	20	\$188*
	10%				15%	inf	\$300*
2.2(0,-)		-5%			10%	5	\$ 91*
		-5%			10%	10	\$ 83*
		-5%			10%	20	\$ 74*
		-5%			10%	inf	\$ 67*
2.3(+,0)			10%		10%	5	\$ 83
			10%		10%	10	\$ 68
			10%		10%	20	\$ 47
			10%		10%	inf	\$ 0
2.4(-,0)				-5%	12%	5	\$109
				-5%	12%	10	\$119
				-5%	12%	20	\$132
				-5%	12%	inf	\$142
3.1b(+,+)	10%		5%		15%	5	\$109
	10%		5%		15%	10	\$120
	10%		5%		15%	20	\$141
	10%		5%		15%	inf	\$200
3.1b(+,+)	5%		10%		15%	5	\$ 92
	5%		10%		15%	10	\$ 83
	5%		10%		15%	20	\$ 71
	5%		10%		15%	inf	\$ 50
4(-,+)	10%			-5%	12%	5	\$131
	10%			-5%	12%	10	\$174
	10%			-5%	12%	20	\$267
	10%			-5%	12%	inf	\$850
5.1(+,-)		-5%	10%		10%	5	\$ 76
		-5%	10%		10%	10	\$ 56
		-5%	10%		10%	20	\$ 35
		-5%	10%		10%	inf	\$ 0
5.2(-,-)		-8%		-5%	7%	5	\$106
		-8%		-5%	7%	10	\$112
		-8%		-5%	7%	20	\$119
		-8%		-5%	7%	inf	\$125

* This is the constant level of spending, since costs are expected to remain constant.