On the Breadth of Earth's Shadow Of Lunar Eclipse - A New Approach To Students' Understanding Of Aristarchus's "Hypothesis 5"

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ABSTRACT

The ancient Greek astronomer Aristarchus was the first astronomer to make a reasonable estimate of the distances of the sun and moon from the earth. In his treatise, "On the Sizes and Distances of the Sun and the Moon", he proposed the "hypothesis 5" saying, "That the breadth of the shadow is two moons" in a lunar eclipse, without any argument. It may be estimated by measuring the size of the arc of Earth's shadow (umbra) projecting on the lunar surface or other means. By studying how students interact with these concepts, we now present a new method to do the evaluation, showing that according to the time of the first contact and the third contact as well as the positions of the two contact points on lunar surface, the ratio of the breadth of Earth's shadow to lunar diameter can be found to have a consistent value of around 2.85. The procedure can be designed to be a middle school science experiment to help students understand the motions of the Earth and Moon.

Keywords: Astronomy Teaching; Lunar Eclipses; Aristarchus

Ancient Greek astronomer Aristarchus's (about 315 BC - 230 BC) treatise *On the Sizes and Distances of the Sun and the Moon* (hereinafter referred to as *On Sizes*) has captured more attention by astronomy educators in recent years. From the hypothesis 1-6, he derived eighteen propositions about the sizes and distances of the Sun and the Moon (Berggren & Sidoli, 2007).

His ingenious ideas could be important references for teaching and research works today. For example, the procedure of finding out the distance of the Moon is quite suitable for middle school science experiment. Although some of the data used by Aristarchus is inaccurate, which leads to a relatively large deviation in the results, the process of his inquiries shows the beauty and elegance of science.

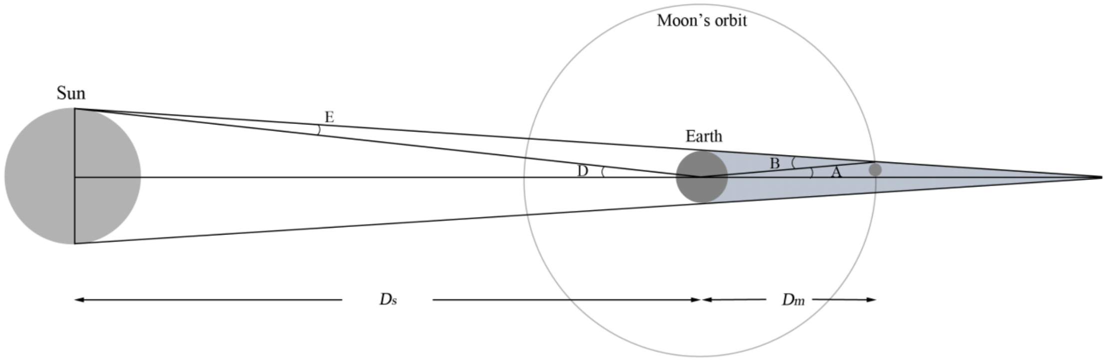
To find the distance to the Moon, he proposed a "hypothesis 5" saying "That the breadth of the shadow is two moons" in a lunar eclipse without any argument. It's common to estimate it by measuring the size of the arc of Earth's shadow projecting on the lunar surface (Cowley, 1989; Sigismondi, 2000). Other methods are proposed in recent years, such as using observations of craters during a lunar eclipse: record the time it takes for the umbral shadow to traverse a crater of known diameter (Lonsdale Hudson, Lu, & McNamara, 2021), and using published light curves of night sky brightness taken during lunar eclipses (Birriela & Adkins, 2019).

In part III of this paper, we introduce a new measurement of the breadth of the Earth's shadow by recording the time interval between the first contact and the third contact as well as the positions of the two contacts on the moon surface. Before that, in the following part II, we briefly summarized Aristarchus's logic of finding the distance to the moon to shows that why the "hypothesis 5" is a crucial parameter in the problem of finding out the distance to the Moon.

ARISTARCHUS'S LOGIC OF FINDING OUT THE DISTANCE TO THE MOON

In the era of Aristarchus, trigonometry and analytical geometry have not yet been invented, and the geometric and arithmetic used in *On sizes* seems being obscure. Scholars had offered various interpretations, and there are some differences in the detail (Ladd, 2005; Mcclung, 2014). Here we brief those interpretations, as described below.

Figure 1. Schematic diagram of finding the distance to the Moon. Based on *On sizes*.



Aristarchus considered about the central lunar eclipse to simplify the derivations. As shown in Figure 1, the moon passes through the center of the Earth's shadow. We designate the radii of the Sun, Earth and Moon as R_s R_e, R_m respectively. The distance to the Sun and the Moon are noted as D_s, D_m, and the radius of Earth's shadow is designated as R_{es}. From Figure 1, we have:

$$< A + < D = < B + < E \quad (1)$$

Noting that A, B, D and E are all small angles (e. g. $< D$ is the angular radius of the Sun, which is around 0.25°, and $< E$ is even smaller; $< B$ is actually the horizon parallax of the Earth, i.e., about 1°, and $< A$ is smaller than it), we got: $< A \approx sin < A = R_{es}/D_m$, $< B \approx sin < B = R_e/D_m$, $< D \approx sin < D = R_s/D_s$, $< E \approx sin < E = R_e/D_s$. Because the angular sizes of the Moon and the Sun are almost equal, we designated them as θ. Then we get:

$$< D = \theta/2cR_s/D_s = R_m/D_m \quad (2)$$

Combining these equations, it can be found that:

$$R_s/R_e = (1 + D_s/D_m)/(1 + R_{es}/R_m) \quad (3)$$

Equation (3) has two important parameters on the right side: 1, the ratio of the Sun's distance to the Moon's distance, and we simplified it as a, $a = D_s/D_m$; 2, the ratio of the breadth of the Earth's shadow to the lunar diameter, and we simplified it as b, $b = R_{es}/R_m$.

We want to find out the distance to the Moon (D_m). That says, we should find out the expression of D_m/R_e. According to equation (2), we know:

$$D_m = 2R_m/\theta \quad (4)$$

According to equation (3), we get:

$$R_e = R_s/[(1 + a)/(1 + b) \quad (5)$$

So, it can be found that:

$$\frac{D_m}{R_e} = \frac{2R_m(1+a)/(1+b)}{R_s \times \theta}$$

It is known from (2) that $R_m/R_s = D_{m/}D_s = 1/a$, thus:

$$\frac{D_m}{R_e} = \frac{2(1+a)/(1+b)}{a\times\theta}, a = \frac{D_s}{D_m}, b = \frac{R_{es}}{R_m} \quad (6)$$

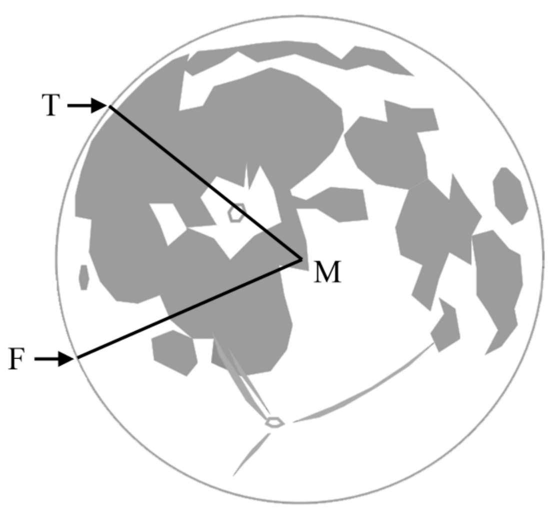
The Earth's radius R_e is found by Eratosthenes, who is almost contemporary with Aristarchus. Therefore, as long as knowing the ratio of the Sun's distance to the Moon's distance a (Aristarchus had already got it through quarter moon), the breadth of Earth's shadow R_{es}, and the angular size of the Moon or the Sun θ, the distance to the moon (D_m) was figured out.

A NEW WAY TO FIND THE BREADTH OF EARTH'S SHADOW

Unlike the value of $D_{s/}D_m$, little attention had been paid to "hypothesis 5" by later scholars. It may be partly because of the "take it for granted" illusion that R_{es}/R_m can be easily found out by measuring the arc of Earth's shadow casting on the lunar surface. But this is found to be a relatively rough approach because of the vague of the shadow arc and the subjective fitting of it. For example, Costantino found a value being $2.80R_m$ to $3.56R_m$ (Sigismondi, 2000). Moreover, Lonsdale et al. (2021) gave an estimation of $2.31R_m$ to $2.78R_m$ by capturing images of the whole Moon as it entered and left the Earth's umbra and then drew tangents to the edge of the umbra to estimate its radius.

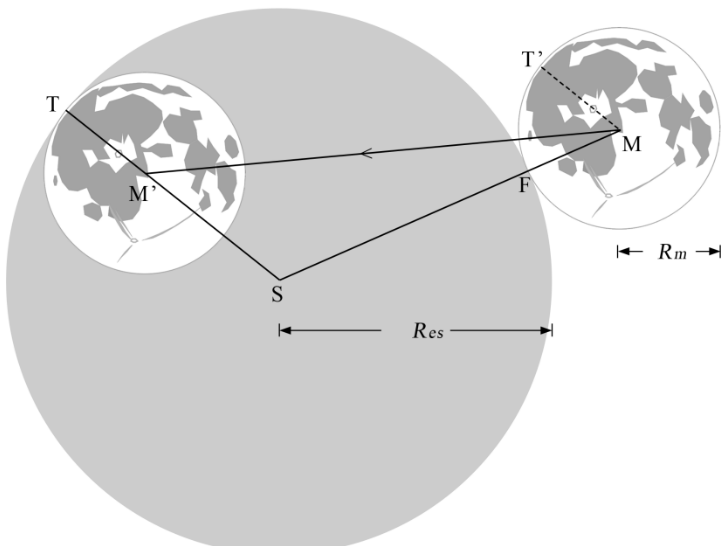
Here we proposed another geometric method by measuring the time interval between the first contact and the third contact as well as the positions of the two contacts on lunar surface. As illustrated in Figure 2, assuming that the Moon first contacts with the umbra of Earth at point F (on moon surface). After the total eclipse, it lights again and makes a third contact with the umbra at point T. The time interval of the two contacts is *t* hours.

Figure 2. Assuming that in a total lunar eclipse, the first contact occurs at F on the lunar surface and the third contact occurs at T. M is the center of the surface. The time interval between the two contacts is *t* hours.



Then, according to the positions of F and T, students are able to draw the path of the Moon during the two contacts, as shown in Figure 3. We ignored the movement of the earth for the moment (in the next part, we'll give a brief discussion about the affection of the motion of the Earth) for the following reasons. The angular velocity of Earth's shadow (relative to the background stars) is equal to the angular velocity of the revolution of the Earth, which is about 360° dividing by one year. In contrast, the angular velocity of the revolution of the Moon approximates 360° dividing by one month (strictly, sidereal month), which is about 13 times of that of Earth's shadow. The time interval of the two contacts is usually about one to three hours, and the angular displacement of Earth's shadow should be less than 0.12°, which is usually much less than the shadow's breadth.

Figure 3. The movement of the Moon from first contact to third contact. The larger circle represents the shadow of the Earth, and it is approximately unmoved during the two contacts (the time interval of the two contacts is usually about one to three hours, and the Earth's shadow moves an angle less than 0.12°, which is much less than the Moon). The center of the Moon moves from M to M' and the center of the shadow S remains unchanged. MT' is parallel to M'T.



In Figure 3, $R_{es} = SF$ is the angular radius of Earth's shadow. The angular radius of the Moon is R_m, and $R_m = MF$ $M'S = R_{es} - R_m$, $SF = R_{es} + R_m$. In the triangle M'MS, we have:

$$M'M^2 = (R_{es} - R_m)^2 + (R_{es} + R_m)^2 - 2(R_{es} - R_m)(R_{es} - R_m)cos < M'SM \quad (7)$$

Then, we get:

$$\frac{R_{es}}{R_m} = \sqrt{\frac{M'M^2/R_m^2 - 2(1 + cos < M'SM}{2(1 - cos < M'SM}} \quad (8)$$

MT' is parallel to $M'T$, thus $< M'SM = 180^0 - < FMT$ is exactly the $< FMT$ in Figure 2 and can be measured from it. What's more, we know that the length of sidereal month is 27.32 days (Karttunen, 2017), so the angular velocity of the Moon is: 360°/27.32 days = 0.549°/hour. Neglecting the movement of the Earth, and knowing the time interval between the first contact and the third contact is *t* hours, we can find out that M'M=0.549°/hour × *t* hour = 0.549°×*t*. Finally, the angular radius of the Moon is almost constant, and we may take it as $+R_m$=0.25°. Substituting these parameters, the equation (8) is simplified to:

$$\frac{R_{es}}{R_m} = \sqrt{\frac{2.41*t^2+\cos\angle FMT-1}{1+cos\angle FMT}} \quad (9)$$

Once we know the time interval t and the positions of the first and second contacts, R_{es}/R_m can be found, and the distance to the Moon can be derived by equation (6). We examined this method through the total lunar eclipse occurred on January 31, 2018 (Espenak, 2013). The first contact occurred on 11:48:27 UT, the third contact occurred on 14:07:51 UT, and the time interval is t = 2.32h. The angle ∠FMT was measured to be about: ∠FMT = 56°±1°. Hence, we got the result:

$$\frac{R_{es}}{R_m} = \sqrt{\frac{2.41 * t^2 + \cos(56^0 \pm 1^0) - 1}{1 + cos(56^0 \pm 1^0)}}$$

Substituting $\frac{R_{es}}{R_m} = 2.83, a \approx 400$, and $\theta = 2r = 0.5^0$ into equation (6), the moon's distance D_m is found out to be $59.8R_e$. It's coincidence with the actual value which varies between $57.5R_e$ and $63.7R_e$.

More lunar eclipses have been checked and the results are listed in table 1. It shows that the calculated R_{es}/R_m are coincident but systematic larger than the actual values. This is because we ignored the orbital motion of the Earth. Noting that the maximum relative error of the four R_{es}/R_m is less than 5% (2022 May 16), the method proposed here seems effective.

Table 1. R_{es}/R_m (the last line) obtained through four lunar eclipses. The eclipse data, i.e. the first five lines, are token from NASA (Espenak, 2013) and ∠FMT is measured from NASA data. The uncertainty of R_{es}/R_m is introduced by the measurement of ∠FMT.

Date of lunar eclipse	First contact (UT)	Third contact (UT)	R_{es}/R_m	Time interval (t, hours)	∠FMT	R_{es}/R_m (eq. 9)
2018 Jan 31	11:48:27	14:07:51	2.74	2.32	56°±1°	2.83±0.01
2019 Jan 21	03:33:54	05:43:16	2.74	2.17	72°±1°	2.85±0.02
2021 May 26	09:44:57	11:25:55	2.77	1.68	108°±1°	2.83±0.03
2022 May 16	02:27:53	04:53:56	2.76	2.43	47°±1°	2.88±0.01

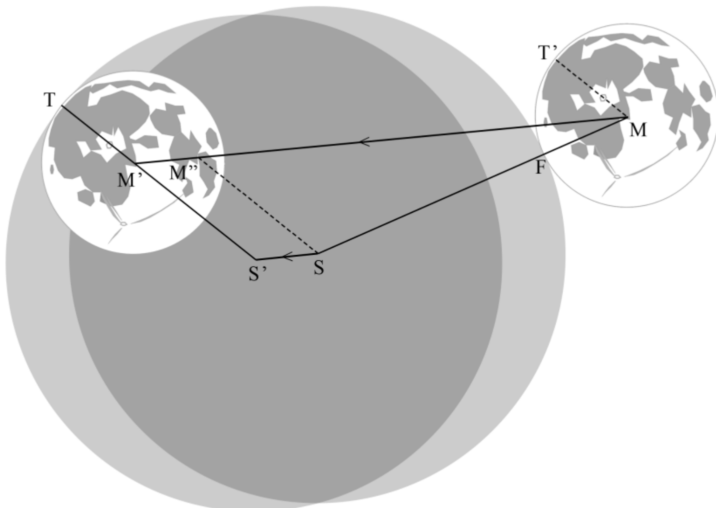
DISCUSSION

In practice, timing the two contacts is relatively straightforward and does not introduce too many errors to a final result. But there is no way to guarantee the accuracy of designating the two contact points with naked eye observations. As such, it would seem on that basis that Aristarchus's "hypothesis 5" was unlikely made by this means. But in nowadays, we can use video cameras to find out the exact contact points, and thus land on a much more accurate result than Aristarchus did.

One might further think about that there are four contacts in a total lunar eclipse, and theoretically, we can use any two of them to do the *time* and *angle* measurements. For example, one could record the time interval between the second contact and the third contact as well as the angle given by them, and then derive a corresponding equation (it would be different with equation 9) to obtain the value of R_{es}/R_m, just like what we did in the previous part. In practice, it is recommended to record all the needed information of these four contacts. In other words, the times and the positions on the moon surface of all the four contacts and derive more than one R_{es}/R_m, and then take an average to reduce the measurement errors. But we'd better be careful about the pair of contacts of the first and fourth, because the time interval between them maybe so long that the Earth's orbital motion would lead to a larger error.

As for the affection of the movement of the Earth, it would be a little bit complicated to give a complete analysis. We may simplify the discussion by neglecting the obliquity of the moon path, which is about 5°09'.

Figure 4. Neglecting the obliquity of the moon path, the Earth's shadow moves parallel to the moon from S to S' in the time interval *t*. MM' is the path of the Moon from the first contact to the third contact. SM" is an auxiliary line parallel to S'M'. The displacement of the Earth's shadow is dramatically enlarged here.



As illustrated in Figure 4, assuming that the Earth's shadow moves parallel to the moon during a lunar eclipse and noting that it moves in the same direction as the moon, the center of the shadow would move from S to S' in the time interval *t*. SM" is an auxiliary line parallel to S'M', and M'M"=S'S. Hence, the equation (8) should be rewritten as:

$$\frac{R_{es}}{R_m} = \sqrt{\frac{(M'M-S'S)^2/R_m^2-2(1+cos<M''SM}{2(1-cos<M''SM}} \tag{10}$$

where S'S equals to the angular distance that the Earth moves during time *t*, and is approximate to: S'S = *t* ×360°/365 days = 0.041° × *t*. Just like the ∠M'SM in equation (8), ∠M"SM = 180° - ∠FMT' = 180° - ∠FMT. Taking the angular radius of the moon R_m as 0.25° and M'M = 0.549°×*t*, equation (10) turns into:

$$\frac{R_{es}}{R_m} = \sqrt{\frac{2.06*t^2+cos<FMT-1}{1+cos<FMT}} \tag{11}$$

For the total lunar eclipse occurred on 2018 Jan 31, now we get $R_{es}/R_m = 2.61 \pm 0.01$ through equation 11, which is smaller than the previous result. $R_{es}/R_m =$ values derived from the other eclipses are listed in Table 2. The relative errors of $R_{es}/R_m =$vary from 4% (2022 May 16) to 8% (2021 May 26) and are overall slightly larger than the previous results. This indicates the complexity of the motions of Earth and Moon. In middle school astronomy practice, it would be more concise to find $R_{es}/R_m =$from equation (8) or (9).

Table 2. The resulted R_{es}/R_m from equation 11 (the last line), which take into account the orbit motion of the Earth with a zero obliquity of the moon path. The other lines are the same as in table 1.

Date of lunar eclipse	R_{es}/R_m	Time interval (t, hours)	<FMT	R_{es}/R_m (eq. 9)	R_{es}/R_m (eq. 11)
2018 Jan 31	2.74	2.32	56°±1°	2.83±0.01	2.61±0.01
2019 Jan 21	2.74	2.17	72°±1°	2.85±0.02	2.62±0.02
2021 May 26	2.77	1.68	108°±1°	2.83±0.03	2.56±0.03
2022 May 16	2.76	2.43	47°±1°	2.88±0.01	2.66±0.01

Not only is a total lunar eclipse extremely spectacular, but also it has been seen much more often than total solar eclipses. On average, a total lunar eclipse is seen almost twice every five years in the same place. It's a free astronomy lesson given by the nature, and more than billions of "students" attend it! Exploring the mysteries of the nature by eclipse observations, even if the work of our predecessors is repeated, there may be new discoveries.

AUTHOR BIOGRAPHY

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NOTES