A Generalization Of Lattice Specifications For Currency Options

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Abstract

This article revisits the topic of two-state pricing of currency options. It examines the models developed by Cox, Ross, and Rubinstein, Rendleman and Bartter, and Trigeorgis, and presents two alternative binomial models based on the continuous and discrete time Geometric Brownian Motion processes respectively. This work generalizes the standard binomial approach incorporating the main existing models as particular cases. The proposed models are straightforward, flexible, accommodate any drift condition and afford additional insights into binomial trees and lattice models in general. Further, the alternative parameterizations are free of the negative aspects associated with the Cox, Ross, and Rubinstein model.

Introduction

After the seminal article by Black and Scholes (BS) (1973), several methods for valuing derivative securities have been proposed. In Merton (1973) the BS model is extended to include valuing an option on a stock, or index, that pays continuous dividends. Feiger and Jaquillat (1979) and, later, Garman and Kohlhagen (1983) and Biger and Hull (1983) extended the BS model to value currency options. Barone-Adesi and Whaley (1987) utilized a quadratic approximation approach to extend the BS framework to the valuation of American options. Cox, Ross, and Rubinstein (CRR) (1979), and Rendleman and Bartter (RB) (1979) pioneered the two-state lattice approach, which is a powerful tool that can be used to value a wide variety of contingent claims. In the binomial setting, valuation by arbitrage arguments is clear. This technique is based on the formation of a risk-free portfolio by combining a currency and an option on this currency. Such a portfolio should earn the risk-free interest rate. The idea was not new and its genesis is the partial differential equation developed by BS. With specific parameters, CRR and RB show that option values from their flexible binomial models converge to the adjusted "celebrated BS model" values described in Feiger and Jaquillat.

The CRR and RB methodology has been later used, slightly modified or extended by, among others, Hull and White (1998), and Boyle (1986). Tian (1993) proposed an alternative binomial model and compared its performance to that of the CRR model. However, the results were later criticized in Easton (1996). Trigeorgis (1991) delineated a log-transformed variation of the binomial model, which supposedly overcomes the problems of consistency, stability and efficiency associated with the CRR specification and various other numerical techniques. According to Trigeorgis (1991, pp. 319), this methodology relative to that of CRR's "compares favorably in terms of computational efficiency due to the log-transformation."

This article revisits two-state option pricing and presents two alternative models based on continuous and discrete time Geometric Brownian Motion processes respectively. The proposed methodology proves extremely flexible as it accommodates any centering condition. This flexibility is achieved by solving for two parameters as a function of the third. Additionally, the RB model is extended and the log-transformed parameterization suggested by Trigeorgis is shown to be mathematically identical to a particular case of the extended RB model. Thus the enhanced computational efficiency attributed to the log-transformation proves mistaken, as it is the result of an exact solution and the specified centering condition.

The main contribution of this work is pedagogical in nature. The proposed parameterizations are simple and flexible alternatives to the popular existing specifications and afford additional insights into binomial trees and lattice models in general.

The format of this paper is as follows: the first section swiftly reviews the continuous time Geometric Brownian Motion (GBM) process; the second presents the RB and CRR models; the third discusses the continuous time GBM process and presents the alternative binomial model (the "ABMC"); the fourth provides the discrete time GBM process and presents the alternative binomial model (the "ABMD"); the fifth reviews the log-transformed binomial model and proves this to be a particular case of an extended RB parameterization; the sixth revisits binomial models, extends the RB model to accommodate numerous centering conditions and compares it to the ABMC and ABMD models; the seventh provides the conclusions.

Continuous Time Geometric Brownian Motion

In a risk-neutral world, if one assumes that the spot exchange rate *S* follows a continuous-time Geometric Brownian Motion process (GBM) then:

$$dS = \tilde{r}Sdt + \sigma Sdz \tag{1}$$

where: $\tilde{r} = r - r_f$,

r is the instantaneous domestic risk-free interest rate,

r_f is the instantaneous foreign risk-free interest rate,

σ is the instantaneous volatility of the spot exchange rate,

dt is an infinitely small increment of time, and

dz is a Wiener process.

By using Ito's Lemma, one can show:

$$d\ln(S) = \left(\tilde{r} - \frac{\sigma^2}{2}\right)dt + \sigma dz \tag{2}$$

or:

$$dX = \alpha dt + \sigma dz \tag{3}$$

where $X = \ln(S)$, and $\alpha = \left(\tilde{r} - \frac{\sigma^2}{2}\right)$. As a result, $\ln(S)$ follows a generalized Wiener process for the time period $(0, t)$, where t is a point in time, and the variable $\hat{X} = X_t - X_0 = \ln\left(\frac{S_t}{S_0}\right)$ is normally distributed with a mean of $\alpha \cdot t$ and a variance of $\sigma^2 t$, and S_0 and S_t represent the spot exchange rate at time 0 and t respectively. More succinctly:

$$\hat{X} \sim N\left(\alpha \cdot t, \sigma\sqrt{t}\right) \tag{4}$$

The continuously compounded rate of return (R) realized during the period $(0, t)$ can be defined by the following equation:

$$S_t = S_0 e^{Rt} \tag{5}$$

accordingly:

$$R = \frac{1}{t}\ln\left(\frac{S_t}{S_0}\right) = \frac{1}{t}\cdot \hat{X} \tag{6}$$

or:

$$t \cdot R = \hat{X} \tag{7}$$

Thus, the continuously compounded rate of return R is normally distributed with a mean of $\alpha = \tilde{r} - \frac{\sigma^2}{2}$ and a variance of $\frac{\sigma^2}{t}$. Taking the expected value of both sides of equation (7) one can obtain:

$$E(t \cdot R) = t \cdot \alpha = E\left(\hat{X}\right) = E\left[\ln\left(\frac{S_t}{S_0}\right)\right] \tag{8}$$

In a binomial model, the spot exchange rate can either move up from S_0 to $u \cdot S_0$ or down to $d \cdot S_0$, where *u* and *d* are two parameters such that *u* is greater than one and *d* is less than one. Since the spot exchange rate follows a binomial model, the variable $\left(\frac{S_t}{S_0}\right)$ has the following distribution:

$$\begin{cases} u & \text{with risk - neutral probability } p \\ d & \text{with risk - neutral probability } (1-p) \end{cases} \tag{9}$$

and for the lattice approach:

$$E\left(\ln\frac{S_t}{S_0}\right) = p\ln(u) + (1-p)\ln(d) \tag{10}$$

or:

$$E\left(\hat{X}\right) = pU + (1-p)D \tag{11}$$

where:

$$U = \ln(u),\ D = \ln(d) \tag{12}$$

Therefore, the variable $\hat{X}$ follows the distribution provided below:

$$\begin{cases} U & \text{with risk - neutral probability } p \\ D & \text{with risk - neutral probability } (1-p) \end{cases} \tag{13}$$

It can be shown that the variance, for the lattice approach, of the variable $\ln\frac{S_t}{S_0}$ is given by the following:

$$Var\left(\ln\frac{S_t}{S_0}\right) = p(1-p)\left[\ln\frac{u}{d}\right]^2 \tag{14}$$

or:

$$Var\left(\hat{X}\right) = p(1-p)\cdot(U-D)^2 \tag{15}$$

The RB and CRR Models

RB (1979) and CRR (1979) proposed the following system:

$$E\left[\ln\frac{S_{\Delta t}}{S_0}\right] = p\ln(u) + (1-p)\ln(d) = \mu\cdot\Delta t \tag{16}$$

$$Var\left[\ln\frac{S_{\Delta t}}{S_0}\right] = p(1-p)\left[\ln(u/d)\right]^2 = \sigma^2\Delta t \tag{17}$$

where Δt is equal to T/n, T is the time to maturity and n is the number of time steps.

RB (1979, p. 1101) suggest the following value for μ:

$$\mu = \frac{\tilde{r} - \frac{1}{2}\sigma\sqrt{T}\left[\sqrt{\frac{1-p}{p}} - \sqrt{\frac{p}{1-p}}\right] - \frac{1}{4}\sigma^2\left[\frac{1-p}{p} + \frac{p}{1-p}\right]}{\left[1 + \left(\sqrt{\frac{1-p}{p}} - \sqrt{\frac{p}{1-p}}\right)\frac{1}{\sqrt{T}}\right]} \tag{18}$$

and indicate that "*the best approximation would occur if* $\mu = \tilde{r} - \frac{\sigma^2}{2}$" which implies that $p = \frac{1}{2}$. This should not be an approximation because by substituting Δt for t in equation (8) one can obtain:

$$E\left(\ln\frac{S_{\Delta t}}{S_0}\right) = \Delta t\cdot\alpha = \Delta t\cdot\left(\tilde{r} - \frac{\sigma^2}{2}\right) \tag{19}$$

Therefore, in a risk-neutral world $\mu = \tilde{r} - \frac{\sigma^2}{2} = \alpha$ is not just the best approximation; it is the only correct value for equation (16).
RB suggest the following exact solution to the system (16, 17):

$$u = e^{\mu\cdot\Delta t + a\cdot\sigma\sqrt{\Delta t}}, \quad d = e^{\mu\cdot\Delta t - b\cdot\sigma\sqrt{\Delta t}} \tag{20}$$

where $a = \dfrac{(1-p)}{\sqrt{p(1-p)}}$, and $b = \dfrac{p}{\sqrt{p(1-p)}}$. If one sets $p = \dfrac{1}{2}$ then the values *u* and *d* are given by the following[1]:

$$u = e^{\left(\tilde{r} - \frac{\sigma^2}{2}\right)\cdot\Delta t + \sigma\sqrt{\Delta t}}, \quad d = e^{\left(\tilde{r} - \frac{\sigma^2}{2}\right)\cdot\Delta t - \sigma\sqrt{\Delta t}} \tag{21}$$

CRR suggest different parameters as a solution to the system (16, 17):

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}} \tag{22}$$

$$p = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\Delta t} \tag{23}$$

These values for *u*, *d* and *p* from (22, 23) satisfy equation (16) exactly and equation (17) approximately. Substituting the values for *u*, *d* and *p* from (22, 23) into the left-hand side of (17) one obtains the following:

$$\sigma^2 \Delta t \cdot \left(1 - \frac{\mu^2 \Delta t}{\sigma^2}\right) \tag{24}$$

and when terms of higher order than Δt are ignored:

$$\sigma^2 \Delta t \cdot \left(1 - \frac{\mu^2 \Delta t}{\sigma^2}\right) = \sigma^2 \Delta t - \mu^2 (\Delta t)^2 \approx \sigma^2 \Delta t$$

For sufficiently small Δt, equation (17) can be approximately satisfied. However, when $\left(1 - \dfrac{\mu^2 \Delta t}{\sigma^2}\right) < 0$, that is $\Delta t > \dfrac{\sigma^2}{\mu^2}$, the left-hand side of equation (17), which is the variance of the lattice distribution, is negative and equation (17) can not be satisfied. While CRR suggests a value for *p* given by (23), the following value is actually applied:

$$p = \frac{e^{\tilde{r}\Delta t} - d}{u - d} \tag{25}$$

If $\Delta t > \dfrac{\sigma^2}{\tilde{r}^2}$ the CRR model gives negative probabilities because:

$$p = \frac{e^{\tilde{r}\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} > 1$$

thus:

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[1] The property that the risk-neutral probability is equal to one-half is generally attributed to Jarrow and Rudd (1983).

$1 - p < 0$

As direct consequence of the approximate solution to equation (17), for the CRR model, the actual volatility at any node of the lattice is downward biased unless Δt is sufficiently small. This is not the case for the RB model, because if $p = \frac{1}{2}$ then $\mu = \tilde{r} - \frac{\sigma^2}{2}$ and (21) is an exact solution to the system (16, 17). Thus, the RB parameters have the same mean and variance of the underlying lognormal diffusion process for any step size and the CRR parameters result in the same mean for any step size but the same variance only in the limit.

The Alternative Binomial Model for Continuous Time Geometric Brownian Motion (ABMC)

In a risk-neutral world, the expected value and the volatility (δ) of the spot exchange rate at time $t + \Delta t$ are given by (see Appendix):

$$E(S_{t+\Delta t}) = S_t e^{\tilde{r}\Delta t} \tag{26}$$

$$\delta(S_{t+\Delta t}) = S_t \cdot e^{\tilde{r}\cdot\Delta t} \cdot \sqrt{(e^{\sigma^2\cdot\Delta t} - 1)} \tag{27}$$

where *t* is the current time, and Δt is a relatively short period of time. Therefore[2]:

$$\tilde{r}\Delta t = \ln \frac{E(S_{t+\Delta t})}{S_t} = \ln E\left(\frac{S_{t+\Delta t}}{S_t}\right) \tag{28}$$

As a direct consequence of the binomial distribution (9), one obtains the following:

$$E\left(\frac{S_{t+\Delta t}}{S_t}\right) = pu + (1-p)d \tag{29}$$

By using (26) and (29) one can find:

$$pu + (1-p)d = e^{\tilde{r}\Delta t} \tag{30}$$

The central moment of order *n* consistent with distribution (9) is given by:

$$M_n = p(u-m)^n + (1-p)(d-m)^n \tag{31}$$

where *m* denotes the expected value of $\left(\frac{S_{t+\Delta t}}{S_t}\right)$:

$$m = pu + (1-p)d$$

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[2] According to equation (28), $\tilde{r}\Delta t = \ln\left(E\,\frac{S_{t+\Delta t}}{S_t}\right)$. It is tempting to assume that $\tilde{r}\Delta t = E\left(\ln\frac{S_{t+\Delta t}}{S_t}\right)$. However, $\ln\left(E\,\frac{S_{t+\Delta t}}{S_t}\right) \neq E\left(\ln\frac{S_{t+\Delta t}}{S_t}\right) = \Delta t \cdot \left(\tilde{r} - \frac{\sigma^2}{2}\right)$ as a result of equation (8).

By substitution:

$$M_n = p(1-p)[(1-p)^{n-1} + p^{n-1}(-1)^n](u-d)^n$$

thus:

$$M_1 = 0,$$
$$M_2 = p(1-p)(u-d)^2,$$
$$M_3 = p(1-p)(1-2p)(u-d)^3$$

In order to match the variance (*V*) (or volatility (δ)) of the spot exchange rate with the lattice parameters, using the second central moment, one has to satisfy the following equation:

$$M_2 = p(1-p)(u-d)^2 = V\left(\frac{S_{t+\Delta t}}{S_t}\right)$$

or,

$$\sqrt{p(1-p)}(u-d) = \delta\left(\frac{S_{t+\Delta t}}{S_t}\right) = e^{\tilde{r}\cdot\Delta t}\cdot\sqrt{(e^{\sigma^2\cdot\Delta t}-1)} = e^{\tilde{r}\cdot\Delta t}\cdot\tilde{\sigma}\sqrt{\Delta t} \tag{32}$$

where $\tilde{\sigma}\sqrt{\Delta t} = \sqrt{(e^{\sigma^2\cdot\Delta t}-1)}$ [3].

By solving the system (30, 32) with respect to *u* and *d*, one can obtain the following parameters for the ABMC model:

$$u = e^{\tilde{r}\Delta t} + a\cdot e^{\tilde{r}\Delta t}\cdot\tilde{\sigma}\sqrt{\Delta t}, \quad d = e^{\tilde{r}\Delta t} - b\cdot e^{\tilde{r}\Delta t}\cdot\tilde{\sigma}\sqrt{\Delta t}$$

or,

$$u = e^{\tilde{r}\Delta t}\cdot(1+a\cdot\tilde{\sigma}\sqrt{\Delta t}), \quad d = e^{\tilde{r}\Delta t}\cdot(1-b\cdot\tilde{\sigma}\sqrt{\Delta t}) \tag{33}$$

where $a = \dfrac{(1-p)}{\sqrt{p(1-p)}}$, and $b = \dfrac{p}{\sqrt{p(1-p)}}$.

The Alternative Binomial Model for Discrete Time Geometric Brownian Motion (ABMD)

The discrete-time version of GBM process for a sufficiently short period of time, Δt, is given by:

$$\frac{S_{t+\Delta t}-S_t}{S_t} = \frac{\Delta S}{S_t} = \tilde{r}\Delta t + \sigma\vartheta\cdot\sqrt{\Delta t}$$

where ϑ is a random drawing from a standard normal distribution $N(0,1)$. It assumes that the proportional

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[3] If terms of order $(\Delta t)^2$ are ignored then $\tilde{\sigma}\sqrt{\Delta t} = \sqrt{(e^{\sigma^2\cdot\Delta t}-1)} \approx \sigma\sqrt{\Delta t}$.

change in the spot exchange rate $\frac{\Delta S}{S_t}$ is normally distributed with a mean of $\tilde{r}\Delta t$ and standard deviation of $\sigma\sqrt{\Delta t}$, that is, $\frac{S_{t+\Delta t}}{S_t}$ is normally distributed with the mean of $1+\tilde{r}\Delta t$ and the same standard deviation[4] of $\sigma\sqrt{\Delta t}$. Analogous to the case of the ABMC, in order to match the expected value and volatility of the spot exchange rate to the lattice parameters for the ABMD, one has to solve the following system:

$$pu+(1-p)d=1+\tilde{r}\cdot\Delta t \tag{34}$$
$$\sqrt{p(1-p)}(u-d)=\sigma\sqrt{\Delta t} \tag{35}$$

By solving the system (34, 35) with respect to *u* and *d*, we obtain the following parameters for the ABMD model:

$$u=1+\tilde{r}\cdot\Delta t+a\cdot\sigma\sqrt{\Delta t},\quad d=1+\tilde{r}\cdot\Delta t-b\cdot\sigma\sqrt{\Delta t} \tag{36}$$

where $a=\frac{(1-p)}{\sqrt{p(1-p)}}$, and $b=\frac{p}{\sqrt{p(1-p)}}$.

The Log-Transformed Binomial Method

Trigeorgis' log-transformed binomial model is based on equation (11) and (15). For the period of time $\Delta t=\frac{T}{n}$, where T is the time to maturity and n is the number of time steps, the system can be given as follows:

$$E[\Delta X]=pU+(1-p)D=\alpha\cdot\Delta t \tag{37}$$
$$Var[\Delta X]=p(1-p)[U-D]^2=\sigma^2\Delta t \tag{38}$$

where $\Delta X=\ln\left(\frac{S_{\Delta t}}{S_0}\right)$. By solving the system (37, 38) with respect to *U* and *D*, one can obtain the following parameters for the log-transformed binomial model:

$$U=\alpha\cdot\Delta t+a\cdot\sigma\sqrt{\Delta t},\quad D=\alpha\cdot\Delta t-b\cdot\sigma\sqrt{\Delta t} \tag{39}$$

or, according to (12):

$$u=e^{\alpha\cdot\Delta t+a\cdot\sigma\sqrt{\Delta t}},\quad d=e^{\alpha\cdot\Delta t-b\cdot\sigma\sqrt{\Delta t}} \tag{40}$$

where $a=\frac{(1-p)}{\sqrt{p(1-p)}}$, and $b=\frac{p}{\sqrt{p(1-p)}}$. Comparing equations (20) and (40) one can conclude that the log-transformed binomial model is equivalent to the RB model adjusted for the parameter μ ($\mu\equiv\alpha$) and subject to a drift-free constraint. Given the additive nature afforded via the log-transformation, the condition $u\cdot d=1$ is

[4] The discrete time process is lognormal in the limit because the $\ln(1+x)\to x$ as $x\to 0$. Thus as *n* – the number of steps - increases and $\Delta t\to 0$, the discrete time process approximates the lognormal diffusion process.

equivalent to the condition that $U + D = 0$ suggested by Trigeorgis. Thus, in short, the stated "increased numerical efficiency" provided by the log-transformed model is not the result of the alternative specification but the exact solution to the system and the specified centering condition.

Binomial Models Revisited

The values *u* and *d* proposed by RB (21) and CRR (22) are different. The reason for this is that the former is an exact solution of the system (16, 17) and the latter is an approximation. For CRR, negative probabilities and variance of the lattice distribution under specific conditions are the direct result of this approximation.

However, there is no need for approximations to solve the system (16, 17). The drift-free condition,

$$u \cdot d = 1 \tag{41}$$

specified by the CRR model and implied in (22), can be easily replicated by both the extended RB, ABMC and ABMD models. This can be achieved by determining the appropriate value of the probability *p*. To find this value for *p*, one must substitute the values *u* and *d* from (20), (33) and (36) into (41) for the extended RB, ABMC and ABMD models respectively and solve these equations for *p*. After basic mathematical transformations, one can obtain the following:

$$p = \frac{1}{2}\left[1 - \frac{q}{\sqrt{4 + q^2}}\right] \tag{42}$$

where for the RB model:

$$q_{RB} = \frac{(\sigma^2 - 2\tilde{r})\Delta t}{\sigma\sqrt{\Delta t}}$$

for the ABMC model:

$$q_{ABMC} = \frac{(1 + \tilde{\sigma}^2 \Delta t \cdot e^{2\tilde{r}\Delta t})e^{-\tilde{r}\Delta t} - e^{\tilde{r}\Delta t}}{\tilde{\sigma}\sqrt{\Delta t} \cdot e^{\tilde{r}\Delta t}}$$

and for the ABMD model:

$$q_{ABMD} = \frac{(1 + \sigma^2 \Delta t) - (1 + \tilde{r} \cdot \Delta t)^2}{(1 + \tilde{r} \cdot \Delta t) \cdot \sigma\sqrt{\Delta t}}$$

Note that when terms of higher order than Δt are ignored: $q_{ABMC} \approx q_{ABMD} \approx q_{RB}$ and subsequently, all three models have the same probability. Equation (42) gives a unique value for *p*, which cannot be negative or greater than one, and satisfies the drift-free condition (41) for all models considered[5].

Another example of the flexibility of the RB, ABMC and ABMD models is that one may also grow the tree along the forward. One may set:

[5] According to equation (42), $p \in (0,1)$. Moreover, the smaller Δt the closer *p* to 0.5.

$$u \cdot d = e^{2\tilde{r}\Delta t} \tag{43}$$

This can be achieved by determining the appropriate value of the probability *p*. To find this value, one must substitute the values of *u* and *d* from (20), (33) and (36) into (43) for the extended RB, ABMC and ABMD models respectively and solve for *p*. After algebraic manipulation, one obtains the following for the RB model:

$$q_{RB} = \sigma\sqrt{\Delta t}$$

for the ABMC model:

$$q_{ABMC} = \tilde{\sigma}\sqrt{\Delta t}$$

and for the ABMD model:

$$q_{ABMD} = \frac{\sigma\sqrt{\Delta t}}{1+\tilde{r}\cdot\Delta t}$$

Lastly, rather than imposing conditions (41) or (43), one may select a value for *p* that corresponds to the drift parameter of the GBM process, which is consistent with the BS model. This is achieved by setting the third central moment of the lattice distribution equal to zero. This implies $p = \frac{1}{2}$. Subsequently, the RB model has the following two parameters, which are also given by (21):

$$u = e^{\alpha\cdot\Delta t+\sigma\sqrt{\Delta t}}, \quad d = e^{\alpha\cdot\Delta t-\sigma\sqrt{\Delta t}} \tag{44}$$

and:

$$ud = e^{2\cdot\alpha\cdot\Delta t} \tag{45}$$

The values *u* and *d* proposed for the ABMC model for the case $p = \frac{1}{2}$ are given by:

$$u = e^{\tilde{r}\Delta t}\cdot(1+\tilde{\sigma}\sqrt{\Delta t}), \quad d = e^{\tilde{r}\Delta t}\cdot(1-\tilde{\sigma}\sqrt{\Delta t}) \tag{46}$$

and:

$$ud = e^{2\tilde{r}\Delta t}\cdot(1-\tilde{\sigma}^2\Delta t) \tag{47}$$

Finally, the values *u* and *d* proposed for the ABMD model for the case $p = \frac{1}{2}$ are given as follows:

$$u = 1+\tilde{r}\cdot\Delta t+\sigma\sqrt{\Delta t}, \quad d = 1+\tilde{r}\cdot\Delta t-\sigma\sqrt{\Delta t} \tag{48}$$

and:

$$ud = (1+\tilde{r}\cdot\Delta t)^2 - \sigma^2\Delta t \tag{49}$$

Thus, when terms of higher order than Δt are ignored for all models:

$$ud = 1 + 2\left(\tilde{r} - \frac{\sigma^2}{2}\right) \cdot \Delta t = 1 + 2 \cdot \alpha \cdot \Delta t \qquad (50)$$

where $2 \cdot \alpha \cdot \Delta t$ represents the drift for the time period $2\Delta t$. Note that in all of the cases presented, the ABMC, ABMD, RB, and extended RB models will not under any circumstance give negative probabilities. Further, the variance of the lattice distribution, and the local volatility at each node is precise. This is a direct result of the exact solutions proposed and the underlying methodology. Given the system proposed by CRR and RB, one has three unknowns and only two equations. CRR and RB both initially imposed a constraint on the solution by choosing a value for one of the unknowns a priori. Contrary to this, it is obvious that one can first solve the system for two unknowns as a function of the third one and then select a desired value for this unknown.

Conclusions

Two alternative binomial lattice models are developed in this article. While for option pricing purposes, the binomial lattice has been supplanted by more sophisticated techniques, it is still the most widely used device for pedagogical purposes.

It is widely recognized that convergence of a discrete time lattice model to the continuous time Black-Scholes model is ensured as long as in the limit the first two moments of the discrete time process are consistent with the lognormal diffusion process of the continuous time GBM. While the CRR result is asymptotic, this study demonstrates that the specification of the parameters is the result of an unnecessary approximation. For the CRR model, negative probability and downward biased variance are a direct result of the approximate solution to the specified system. It is true that the consequences are of minimal concern, yet an exact solution, which is free of the negative aspects of approximations, is always preferable. It is often argued that the drift free condition is preferred, as it is easier to calculate the hedge parameters from this specification. This article demonstrates that the drift free condition is easily obtainable without approximations. By solving for two parameters – *u* and *d* – as a function of the third – *p* – the methodology employed in this article affords one the flexibility to specify any centering condition.

This study also shows that while RB believed that the "best approximation would occur" if the drift parameter was set equal to the drift of the GBM process, this is the only correct value for their system. The RB model is extended to accommodate numerous centering conditions. Lastly, the log-transformed binomial model delineated by Trigeorgis is shown to be mathematically identical to the case of the extended RB model with the drift parameter set equal to zero. That is, the log-transformation is not the source of increased numerical efficiency as suggested by Trigeorgis but the result of an exact solution and specified centering condition.

In conclusion, it is obvious that the binomial-pricing method has three independent parameters subject to only two constraints, which are the mean and variance converge in the limit to the appropriate diffusion process. This work demonstrates that the proposed alternatives, which are exact solutions based on continuous and discrete time GBM and implemented on a lattice, not only converge to the appropriate diffusion process but, are simple, flexible alternatives, free of the negative aspects associated with the CRR parameterization and theoretically tractable.

Bibliography

1. Barone-Adesi, G., and Whaley, R. E. "Efficient Analytic Approximation of American Option Values." *Journal of Finance,* 42 (1987), 301-320.
2. Biger, N., and Hull, J., "The Valuation of Currency Options." *Financial Management*, 81 (1983), 24-28.
3. Black, F., and Scholes, M. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy,* 81 (1973), 637-659.
4. Boyle, P.P. "Option Valuation Using a Three-jump Process." *International Options Journal,* 3 (1986), 7-12.
5. Cox, J.C., Ross, S., Rubinstein, M. "Option Pricing: A Simplified Approach." *Journal of Financial Eco-*

nomics, 7 (1979), 229-264.
6. Easton, S.A., "A Note on Modified Approaches to Option Pricing." *The Journal of Futures Markets,* 16, No.5 (1996), 585-594.
7. Feiger, G., Jacquillat, B., "Currency Option Bonds, Puts, and Calls on Spot Exchange and the Hedging of Contingent Foreign Earnings", *Journal of Finance,* 34 (1979), 1029-1139.
8. Garman, M. B., Kohlhagen, S. W., "Foreign Currency Option Values", *Journal of International Money and Finance* 2 (1983), 231-237.
9. Hull, J.C., and White, A. "The use of Control Variate Technique in Option-Pricing." *Journal of Financial and Quantitative Analysis,* 23(2) (1988),161-174.
10. Jarrow, R., and Rudd, A. Option Pricing. Homewood, IL: Dow Jones-Irwin Publishing, (1983).
11. Merton, R.C. "Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science,* 4 (1973), 141-183.
12. Rendleman, R., and Bartter, B. "Two-State Option Pricing." *Journal of Finance,* 34 (1979), 1092-1110.
13. Tian, Y., "A Modified Lattice Approach to Option Pricing." *The Journal of Futures Markets,* 13 (1993), 563-577.
14. Trigeorgis, L., "A Log-Transformed Binomial Numerical Analysis Method for Valuing Complex Multi-Option Investments." *Journal of Financial and Quantitative Analysis,* 26(3) (1991), 309-326.

Appendix

Consider a variable *w* that is normally distributed with a mean of zero and a variance of one: $w \sim N(0,1)$, and an arbitrary constant *a*. The expected value of the variable $A = e^{a\cdot w}$ is given by:

$$E(e^{a\cdot w}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{a\cdot w}\cdot e^{-\frac{w^2}{2}}dw = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\frac{a^2}{2}}\cdot e^{-\frac{(w-a)^2}{2}}dw = e^{\frac{a^2}{2}}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{(w-a)^2}{2}}dw \tag{A.1}$$

Substituting $x = w - a$, one can obtain:

$$E(e^{a\cdot w}) = e^{\frac{a^2}{2}}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}dx = e^{\frac{a^2}{2}} \tag{A.2}$$

According to equation (2),

$$d\ln(S) = \alpha dt + \sigma dz \tag{A.3}$$

where $\alpha = \left(\tilde{r} - \frac{\sigma^2}{2}\right)$.

Integrating the previous equation we have,

$$\int_0^T d\ln(S) = \int_0^T \alpha dt + \int_0^T \sigma dz \tag{A.4}$$

equivalently,

$$\ln\left(\frac{S_T}{S_0}\right) = \alpha\cdot T + \sigma\cdot z_T \tag{A.4}$$

and thus,

$$S_T = S_0 e^{\alpha \cdot T + \sigma \cdot z_T} \tag{A.5}$$

Given this, the expected value and variance of the spot exchange rate at the time T are:

$$E(S_T) = S_0 \cdot e^{\alpha \cdot T} \cdot E(e^{\sigma \cdot z_T}) = S_0 \cdot e^{\alpha \cdot T} \cdot E(e^{b \cdot w}) \tag{A.6}$$

$$V(S_T) = E(S_T^2) - [E(S_T)]^2 = S_0^2 \cdot e^{2 \cdot \alpha \cdot T} \cdot E(e^{2 \cdot b \cdot w}) - [E(S_T)]^2 \tag{A.7}$$

where $w = \frac{z_T}{\sqrt{T}}$, $b = \sqrt{T} \cdot \sigma$.

According to (A.2)

$$E(S_T) = S_0 \cdot e^{\alpha \cdot T + \frac{b^2}{2}} = S_0 \cdot e^{\tilde{r} \cdot T} \tag{A.8}$$

$$V(S_T) = S_0^2 \cdot (e^{2 \cdot \alpha \cdot T + 2 \cdot b^2} - e^{2 \cdot \tilde{r} \cdot T}) = S_0^2 \cdot e^{2 \cdot \tilde{r} \cdot T} \cdot (e^{\sigma^2 \cdot T} - 1) \tag{A.9}$$

Thus, the standard deviation of the spot exchange rate at the time T is:

$$\delta(S_T) = S_0 \cdot e^{\tilde{r} \cdot T} \cdot \sqrt{(e^{\sigma^2 \cdot T} - 1)} \tag{A.10}$$